§3.8

Main Concepts

1. Antiderivatives
2. Rules / Toolbox
3. Constants
4. u-Substitution

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1 At the end of §3.6, we noted that if we know $f'(x)$, we can solve for $f(x)$, up to an added constant. Let's introduce some notation for this. If $F(x)$ is a function such that $F'(x) = f(x)$, then we say $F(x)$ is an <u>antiderivative</u> of $f(x)$. We'll write

"$\int f(x)\,dx$" to denote the <u>general antiderivative</u>. Let's see how this works in practice:

Examples:

① Find $\int 2x\,dx$.

We're looking for all functions whose derivative is $2x$. We know $(x^2)' = 2x$ and further that $(x^2 + C)' = 2x$ for any constant C. In fact, from §3.6, this is all the solutions, so $\boxed{\int 2x\,dx = x^2 + C}$

② Find $\int 6x^2\,dx$.

Again, thinking about derivatives backwards gives one antiderivative — $2x^3$. So we get all of them by adding a constant: $\boxed{\int 6x^2\,dx = 2x^3 + C}$

In other words, if $F(x)$ is <u>an</u> antiderivative of $f(x)$, then <u>the</u> general antiderivative of $f(x)$ is given by $\int f(x)\,dx = F(x) + C$.

2 We did the two examples above by "reversing" the process of taking a derivative, but as things get more complicated, we'd like some tools. We know that coefficients are unaffected by derivatives and that the derivative of a sum is the sum of the derivatives. So, these should be true for antiderivatives:

<u>Coefficient Rule</u>: $\int af(x)\,dx = a\int f(x)\,dx$

<u>Sum Rule</u>: $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

We should also be able to reverse the power rule. We know $(x^{n+1})' = (n+1)x^n$, so we see that $\frac{1}{n+1}(x^{n+1})' = x^n$ and $\left(\frac{1}{n+1}x^{n+1}\right)' = x^n$. In other words, $\frac{1}{n+1}x^{n+1}$ is an antiderivative of x^n (as long as $n \neq -1$ so we don't divide by zero).

<u>Power Rule</u>: $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

EXAMPLES: Find $\int f(x)dx$ in each case.

① $f(x) = x^6$

This is directly the power rule: $\int x^6 dx = \frac{1}{7}x^7 + C$

② $f(x) = 3x^8$

Now we'll also use the coefficient rule: $\int 3x^8 dx = 3\int x^8 dx = 3\left(\frac{1}{9}x^9 + C\right) = \frac{1}{3}x^9 + 3C$

③ $f(x) = x^2 + x^5$

Now the sum rule: $\int (x^2+x^5)dx = \int x^2 dx + \int x^5 dx = \left(\frac{1}{3}x^3 + C\right) + \left(\frac{1}{6}x^6 + D\right)$

④ $f(x) = 2x^4 - x^2$

And all together now: $\int (2x^4 - x^2)dx = \int 2x^4 dx + \int -x^2 dx = 2\int x^4 dx - \int x^2 dx$

$= 2\left(\frac{1}{5}x^5 + C\right) - \left(\frac{1}{3}x^3 + D\right) = \frac{2}{5}x^5 - \frac{1}{3}x^3 + 2C - D.$

Notice each antiderivative creates a constant, so we use different letters for each $(C, D, \ldots)$.

Finally, we can "undo" our memorized derivatives

SPECIAL ANTIDERIVATIVES:

$\int \sin(x)dx = -\cos(x) + C$ $\quad$ $\int \cos(x)dx = \sin(x) + C$ $\quad$ $\int \frac{1}{x}dx = \ln|x| + C$

$\int \frac{1}{1+x^2}dx = \arctan(x) + C$ $\quad$ $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$ $\quad$ $\int e^x dx = e^x + C$

Notice that as we split up our antiderivatives, we got more and more constants. But notice also that if C and D are arbitrary constants, so is $C+D$ or $2D$ or $C-D^3$. This helps us simplify some things — namely, instead of writing a new constant every time, we'll just "lump together" all our constants at the end of a problem and write $+C$. The main thing to watch out for is operations that change how constants act. Let's practice.

EXAMPLES: Evaluate each, lumping together constants wherepossible.

① $\int (x^2+x^5)dx$

Like above, we have $\int (x^2+x^5)dx = \frac{1}{3}x^3 + \frac{1}{6}x^6 + C + D$ but $C+D$ is a constant, so we'll rename it to just be C: $\int (x^2+x^5)dx = \frac{1}{3}x^3 + \frac{1}{6}x^6 + C$.

② $\int 3x^8 dx$

Again, as above, this was $\frac{1}{3}x^9 + 3C$, but we can rename $3C \to C$ to get

$\int 3x^8 dx = \frac{1}{3}x^9 + C$

③ $e^{\int 2x dx}$

This is an odd one. The antiderivative is straightforward — $\int 2x\,dx = 2\int x\,dx = 2\left(\frac{1}{2}x^2 + C\right)$

$= x^2 + 2C$ or, after renaming, $\int 2x dx = x^2 + C$. So, our answer is

$e^{\int 2x dx} = e^{x^2+C} = (e^{x^2})(e^C)$ by exponent rules. Now e^C is also just a constant, so I'll call it again C and get $e^{\int 2x dx} = Ce^{x^2}$. Notice that

we get $\cdot C$ instead of $+C$ — this has nothing to do with calculus and everything to do with exponent rules, but it's worth looking out for as we move forward (and especially in §3.9).

4 We have one more tool to discuss, but let's motivate it with an example:

EXAMPLE: Find $\int \cos(3x)\,dx$.

We know $\int \cos(x)\,dx = \sin(x) + C$, so maybe $\sin(3x)$ is an antiderivative. But not quite:

$(\sin(3x))' = \cos(3x)\cdot(3x)' = \cos(3x)\cdot 3 = 3\cos(3x)$ by the Chain Rule. So, we need to account for the extra 3 with a $\frac{1}{3}$:

$\left(\frac{1}{3}\sin(3x)\right)' = \frac{1}{3}\cos(3x)\cdot 3 = \cos(3x)$.

So $\frac{1}{3}\sin(3x)$ is an antiderivative and $\boxed{\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C}$.

How can we make this process — undoing the Chain Rule — more systematic? Remember that there are typically 2 ways to think of the Chain Rule: 1) As a composition $(f\circ g)'(x)$, and 2) as a coordinate change $u=g(x)$ $y=f(u)$. This second approach will help.

STRATEGY (u-Substitution): When evaluating $\int f(x)\,dx$, you may want to simplify it by replacing a piece of it with a single variable. To do this,

① Identify that piece and write $u = g(x)$.

② Differentiate to get $du = g'(x)\,dx$ and solve for $dx = \frac{du}{g'(x)}$.

③ Plug in $u=g(x)$ and $dx = \frac{du}{g'(x)}$ into $\int f(x)\,dx$. Simplify

④ If you have any "x" remaining, use algebra to replace them with "u"s.

⑤ Evaluate the new antiderivative in terms of u.

⑥ Plug in $u=g(x)$ again to get things back in terms of x.

EXAMPLES: Evaluate $\int f(x)\,dx$ in each case.

① $f(x) = \cos(3x)$

This is the same as above: $\int \cos(3x)\,dx$. Now, we use our tool. It would be nice if we had just "x" inside of cosine, instead of "3x." So, let's try $u = 3x$. Then

$du = 3\,dx$, so $dx = \frac{du}{3}$.

Plugging this in gives: $\int \cos(3x)\,dx = \int \cos(u)\cdot\frac{du}{3} = \frac{1}{3}\int \cos(u)\,du = \frac{1}{3}\sin(u) + C$

$= \frac{1}{3}\sin(3x) + C$ as we got before.

② $f(x) = 2x(x^2+7)^3$.

I.e. we want $\int 2x(x^2+7)^3\,dx$. The "inside function" x^2+7 would be nice to be simpler, so try $u = x^2+7$. Then $du = 2x\,dx$ so $dx = \frac{du}{2x}$, and:

$$\int 2x(x^2+7)^3\,dx = \int 2x(u)^3\,\frac{du}{2x} = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(x^2+7)^4 + C.$$

Notice that the substitution helped in two ways — simplifying the composition $(x^2+7)^3 \longrightarrow u^3$ and addressing the extra factor of $2x$.

③ $f(x) = 3x^2\sqrt{x^3+5}$

Immediately, let's take $u = x^3+5$ so $du = 3x^2\,dx$ and $dx = \frac{du}{3x^2}$. Then

$$\int f(x)\,dx = \int 3x^2\sqrt{x^3+5}\,dx = \int 3x^2\sqrt{u}\,\frac{du}{3x^2} = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}(x^3+5)^{3/2} + C.$$

④ $f(x) = \sin^2(x)\cos(x)\,dx.$

Let's try $u = \sin(x)$, so $du = \cos(x)\,dx$. Then

$$\int f(x)\,dx = \int \sin^2(x)\cos(x)\,dx = \int u^2\cos(x)\,\frac{du}{\cos(x)} = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C.$$

NOTE: There are some general tips to choosing u.

① Look for "inside functions" like x^2+7 from example ②.

② Look for choices of u that make du cause cancellation, like $du = \cos(x)\,dx$ in example ④.

However, there isn't a "right way" to do these problems. In example ④, choosing $u = \sin^2(x)$ or $u = \cos(x)$ would have also worked, but with more algebra and some trig identities. Don't be afraid to just try things and be ready to cross off some work that leads to dead-ends or difficult algebra. The best way to learn is to practice, try, fail, reflect, and repeat!