§3.9

MAIN CONCEPTS

[1] Differential Equations and Initial Conditions

[2] Separable Equations

---

[1] Antiderivatives allow us to answer the question— "If I know the rate at which a quantity changes, then do I know the quantity?" — with the answer of yes, at least up to a constant. Questions like these are examples of differential equations, relationships between an unknown function and its derivatives.

A specific type of differential equation, called an initial value problem, is the main one we study. These problems fill in the last gap — the $+C$ coming from the antiderivative — by providing one additional piece of data, called the initial condition.

EXAMPLES: Find a function $f(x)$ satisfying the following.

① $f'(x) = 2x+1$ and $f(3) = 7$

The first part is the differential equation, and the second is the extra data (initial condition).

Taking antiderivatives of the diff. eq. gives:

$$\int f'(x)\,dx = \int (2x+1)\,dx \quad \rightarrow \quad f(x) = x^2 + x + C$$

[Notice I didn't write $+C$ on both sides. See §3.8, section [3] for a reminder of why].

But now, we can continue by using the initial condition. Plug in $x=3$:

$$f(3) = (3)^2 + (3) + C \quad \longrightarrow \quad 7 = 9+3+C \quad \rightarrow \quad C = 7-12 = -5$$

So, finally, we get $\boxed{f(x) = x^2+x-5}$

② $f'(x) = 2\cos(2x) + 1$ $\qquad f(0) = 10$

Same idea:

$$f(x) = \int f'(x)\,dx = \int (2\cos(2x)+1)\,dx = \sin(2x) + x + C$$

and plugging in $x=0$ gives

$$10 = f(0) = \sin(2(0)) + (0) + C = 0+0+C = C \qquad \text{so} \quad C = 10 \quad \text{and}$$

$$f(x) = \sin(2x) + x + 10.$$

In fact, we can even include more derivatives if we want. However, each derivative loses 1 piece of information (the next constant term), so if we have a second derivative $f''(x)$, then we'll need two initial conditions to put it back together.

EXAMPLE: Find $f(x)$ if $f''(x) = 6x$, $f'(1) = 4$, and $f(3) = 5$.

We start by taking antiderivatives of our differential equation:

$\int f''(x)dx = \int 6x\,dx \quad \rightarrow \quad f'(x) = 3x^2 + C$

Now, my first initial condition can help me solve for C. Plug in $x=1$ to get

$f'(1) = 3(1)^2 + C \quad \rightarrow \quad 4 = 3 + C \quad \rightarrow \quad C = 1$

So $f'(x) = 3x^2 + 1$. Now we repeat:

$\int f'(x)dx = \int (3x^2+1)dx \quad \rightarrow \quad f(x) = x^3 + x + D$

[I've used D to not confuse myself with previous work.]. Now evaluate at $x=3$:

$f(3) = (3)^3 + (3) + D \quad \rightarrow \quad 5 = 27 + 3 + D \quad \rightarrow \quad D = 5 - 30 = -25.$

So $\boxed{f(x) = x^3 + x - 25}$

[2] The differential equations above are "simple" in the sense that the right-hand-side only involved the independent variable x. In many real-world examples, this is unrealistic. For example, if $f(x)$ is the population of fish in a certain lake on day x, then $f'(x)$ probably also depends on $f(x)$ — more fish means more mating means $f'(x)$ is bigger, and too many fish means competition means $f'(x)$ is smaller. Such differential equations can get quite complicated, so we'll study just some of them: separable differential equations.

In a separable diff. eq., the idea is that the two variables involved can be separated on either side of the equals sign, and handled independently. For these problems, we'll typically use Leibniz notation ($\frac{dy}{dx}$) instead of Newton's ($f'(x)$), or switch to it asap.

EXAMPLES: Solve for $f(x)$

① $f'(x) = f(x)$ and $f(0) = 3$.

Notice that $f(x)$ appears on the right so we hope it's separable and switch to Leibniz notation: $y = f(x)$ and so $\frac{dy}{dx} = f'(x)$. Then

$\frac{dy}{dx} = y \quad \rightarrow \quad \frac{dy}{y} = dx$ by multiplying by dx and dividing by y.

We've successfully separated our variables. So now, we take antiderivatives:

$\int \frac{dy}{y} = \int dx \quad \rightarrow \ln(y) = x + C \quad \rightarrow \quad y = e^{x+C} = e^x \cdot e^C = Ae^x$

where we renamed $A = e^C$. I.e. in the original notation, $f(x) = Ae^x$. Now use $x=0$:

$f(0) = Ae^0 \quad \rightarrow \quad 3 = A(1) \quad \rightarrow \quad A = 3.$

So $\boxed{f(x) = 3e^x}$

② $f'(x) = \frac{2x}{3f(x)^2}$ and $f(4) = 2$

Again, let's use x & y: $\frac{dy}{dx} = \frac{2x}{3y^2} \quad \rightarrow \quad 3y^2 dy = 2x\,dx \quad \rightarrow$

$\rightarrow \quad \int 3y^2 dy = \int 2x\,dx \quad \rightarrow \quad y^3 = x^2 + C,$

When $x = 4$, $y = f(4) = 2$, so:

$(2)^3 = (4)^2 + C \longrightarrow 8 = 16 + C \longrightarrow C = -8$.

So $y^3 = x^2 - 8 \rightarrow y = \sqrt[3]{x^2 - 8}$, i.e. $\boxed{f(x) = \sqrt[3]{x^2 - 8}}$