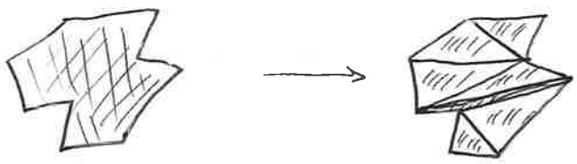
§4.1

MAIN CONCEPTS

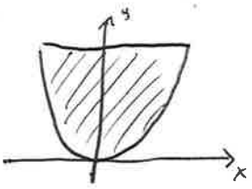
1. Summation
2. Area under a curve

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1. We're shifting gears away from derivatives, at least for a moment. Instead, we'd like to consider a (relatively old) problem: How can we measure the area of a curved shape? In other words, when shapes are made up of straight lines, we can cut it up into triangles, use $A = \frac{1}{2}bh$ for each triangle, and add it up.



But what about the area of this "shield"-like shape, bounded by the parabola $y = x^2$ and the line $y = 4$?




To begin answering this question, we first establish some notation and language around summations, the process of adding up many things.

**Notation:** We'll write $\sum_{n=1}^{N} a_n$ for the expression $a_1 + a_2 + \cdots + a_{N-1} + a_N$. This is called "Sigma notation" ($\Sigma$ is the greek letter capital sigma).

**EXAMPLES:** Compute each sum.

① $\sum_{n=1}^{4} (n+1)$

Following our notation, this is asking us to add up "$n+1$" when $n=1$ then $n=2$, $n=3$, and $n=4$. I.e.

$$\sum_{n=1}^{4} (n+1) = (1+1) + (2+1) + (3+1) + (4+1) = 2+3+4+5 = 14.$$

② $\sum_{n=1}^{3} n^3$

Same ideas plug in $n=1, 2, 3$:

$$\sum_{n=1}^{3} n^3 = (1)^3 + (2)^3 + (3)^3 = 1 + 8 + 27 = 36.$$

**EXAMPLES:** Write each sum in sigma notation.

① $1+2+3+4+\cdots+100$

There are multiple ways, but likely the simplest is $\sum_{n=1}^{100} n$. [$\sum_{n=0}^{99} (n+1)$ and $\sum_{i=1}^{100} i$ also work, for example.]

② $3^2 + 4^2 + 5^2 + \cdots + 11^2$.

Again, a few options: $\sum_{n=3}^{11} n^2$ is probably clearest, $\sum_{n=1}^{9} (n+2)^2$ starts at $n=1$ which is nice, and $\sum_{n=0}^{8} (n+3)^2$ starts at $n=0$ which is also nice.

In order to evaluate these sums (without adding up each term by hand), we need some rules. The first is factoring: we know $3(6)^2 + 3(7)^2 + \cdots + 3(23)^2 = 3\left[6^2 + 7^2 + \cdots + 23^2\right]$, and

this works in general:

DISTRIBUTIVE PROPERTY: $\sum_{n=1}^{N} c \cdot a_n = c \sum_{n=1}^{N} a_n$.

We can also add in different orders: $(1^2+1)+(2^2+2)+(3^2+3)+\cdots+(27^2+27)$
$= (1^2+2^2+\cdots+27^2) + (1+2+\cdots+27)$. I.e. we have

REARRANGEMENT: $\sum_{n=1}^{N}(a_n+b_n) = \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n$.

Finally, some well-known sums (we won't prove these, but feel free to try a few and check!)

POWER SOMS:
- $\sum_{n=1}^{N} 1 = N$
- $\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$
- $\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$
- $\sum_{n=1}^{N} n^3 = \left[\frac{N(N+1)}{2}\right]^2$.

EXAMPLES: Compute each sum.

① $\sum_{n=1}^{20}(2n+1)$

Using our rules: $\sum_{n=1}^{20}(2n+1) = \sum_{n=1}^{20}(2n) + \sum_{n=1}^{20} 1 = 2\sum_{n=1}^{20} n + (20) = 2\cdot\frac{20(20+1)}{2} + 20 = 440$

② $\sum_{n=1}^{7}(n^3+n^2-n-4)$ *

Ditto: $= \sum_{n=1}^{7} n^3 + \sum_{n=1}^{7} n^2 - \sum_{n=1}^{7} n - 4\sum_{n=1}^{7} 1 = \left(\frac{7\cdot 8}{2}\right)^2 + \left(\frac{7\cdot 8\cdot 15}{6}\right) - \left(\frac{7\cdot 8}{2}\right) - 4(7)$

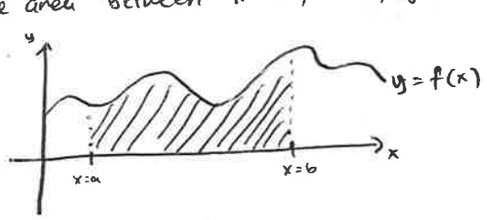
$= 28^2 + 280 - 28 - 28 = 1008$

③ $3+4+5+\cdots+99+100$

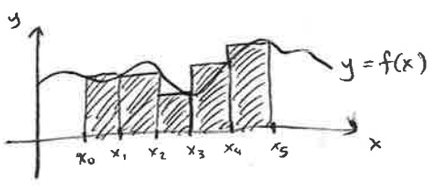
To use the rules, we'll write it in Sigma notation: $\sum_{n=1}^{98}(n+2) = \sum_{n=1}^{98} n + 2\sum_{n=1}^{98} 1$

$= \frac{98\cdot 99}{2} + 2\cdot 98 = 5047$

2 What does all this have to do with area? Well, our idea is to first approximate the area of a curved region by something easier to calculate. To be more concrete, let's say we have a function $f(x)$ and we want to compute the area between $x=a$, $x=b$, $y=f(x)$, and $y=0$.




I.e. this area. To approximate it, we can cut up the domain into smaller pieces, and say that each subregion is approximately a rectangle. Then, the area under $f(x)$ is roughly equal to the sum of the areas of the rectangles!



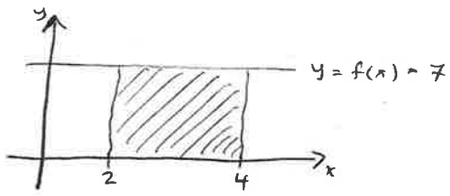

Let's add some variables to make this clearer: If we divide $[a,b]$ into $N$ equal pieces, then each piece has width $\Delta x = \frac{b-a}{N}$. We can also label the endpoints — $x_0 = a$, $x_1 = a+\Delta x$, $x_2 = a+2\Delta x, \ldots, x_N = a+N\Delta x = b$. I.e. $x_n = a + n\Delta x$. Then, for the rectangle between $x_{n-1}$ and $x_n$, we need to pick a height. In the diagram above, we used the right endpoint for the height, so height $= y = f(x_n)$. Thus, the $n^{th}$ rectangle has area

(base) × (height) = $(\Delta x)(f(x_n))$, so the total area is calculated by adding these up:

**AREA ESTIMATE:** Area $\approx \sum_{n=1}^{N} f(x_n)\Delta x$ (where $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$).

**EXAMPLES:** Estimate the area under $f(x)$ between $x=a$ and $x=b$. If possible, also compute the area exactly.

① $f(x) = 7$ on $[2,4]$ with $N = 10$.



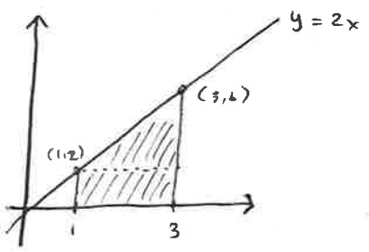

Let's plug everything in: $\Delta x = \frac{b-a}{N} = \frac{4-2}{10} = \frac{1}{5}$.

$x_n = a + n\Delta x = 2 + n \cdot \frac{1}{5} = 2 + \frac{n}{5}$.

Then Area $\approx \sum_{n=1}^{10} f(x_n)\Delta x = \sum_{n=1}^{10} f(2+\frac{n}{5}) \cdot \frac{1}{5} = \sum_{n=1}^{10} (7) \cdot \frac{1}{5} = \frac{7}{5}\sum_{n=1}^{10} 1 = \frac{7}{5}(10) = 14$.

In fact, from the graph we see the area is a rectangle with base = 2, height = 7, so area = $2 \cdot 7 = 14$.

② $f(x) = 2x$ on $[1,3]$ with $N = 8$.




Ditto: $\Delta x = \frac{b-a}{N} = \frac{3-1}{8} = \frac{1}{4}$

$x_n = a + n\Delta x = 1 + n \cdot \frac{1}{4} = 1 + \frac{1}{4}n$

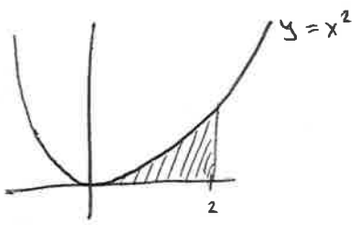
Then Area $\approx \sum_{n=1}^{8} f(x_n)\Delta x = \sum_{n=1}^{8} 2(1+\frac{1}{4}n)(\frac{1}{4})$

$= \sum_{n=1}^{8} \frac{2}{4} + \sum_{n=1}^{8} \frac{2}{16}n = \frac{1}{2}(8) + \frac{1}{8}\left(\frac{8 \cdot 9}{2}\right) = \frac{8}{2} + \frac{9}{2} = \frac{17}{2} = 8.5$

From the picture, we see the actual region is rectangle + triangle, so we can find the area exactly: $A = (\text{base}) \times (\text{height}) + \frac{1}{2}(\text{base})(\text{height}) = (2)(2) + \frac{1}{2}(2)(4) = 8$.

Our estimate was pretty close (but not perfect).

③ $f(x) = x^2$ on $[0,2]$ with $N = 4$.




$\Delta x = \frac{b-a}{N} = \frac{2-0}{4} = \frac{1}{2}$

$x_n = a + n\Delta x = 0 + n(\frac{1}{2}) = \frac{1}{2}n$

So Area $\approx \sum_{n=1}^{4} (\frac{1}{2}n)^2 \cdot (\frac{1}{2}) = \sum_{n=1}^{4} \frac{1}{8}n^2 = \frac{1}{8} \cdot \frac{(4)(5)(9)}{6} = \frac{15}{4} = 3.75$.

I don't know how to compute this area otherwise, so I can't (yet) compute it exactly.