§4.2

Main Concept

1. Exact area

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1. Our area estimate from §4.1 only estimates the area, but if N is really big, the error should be small. So, we should be able to compute the area exactly by taking $N \to \infty$.

Area Formula: The area under $y=f(x)$ between $x=a$ and $x=b$ is

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\Delta x = \lim_{N\to\infty} \sum_{n=1}^{N} f\left(a+\tfrac{b-a}{N}\right)\cdot\tfrac{b-a}{N} \qquad \left(\text{i.e. } \Delta x = \tfrac{b-a}{N} \text{ and } x_n = a+n\Delta x\right).$$

Examples: Find the exact area under $y=f(x)$ on the specified interval.

① $f(x)=7$ on $[2,4]$.

We did this in §4.1, but with our formula, we have $\Delta x = \frac{4-2}{N} = \frac{2}{N}$ and $x_n = 2+n\cdot\frac{2}{N}$ so

$$A = \lim_{N\to\infty} \sum_{n=1}^{N} f\left(2+n\cdot\tfrac{2}{N}\right)\cdot\tfrac{2}{N} = \lim_{N\to\infty} \sum_{n=1}^{N} (7)\cdot\tfrac{2}{N} = \lim_{N\to\infty} \tfrac{14}{N}(N) = \lim_{N\to\infty} (14) = 14.$$

② $f(x)=2x$ on $[1,3]$.

Now $\Delta x = \frac{3-1}{N} = \frac{2}{N}$ and $x_n = 1+n\cdot\frac{2}{N}$ so

$$A = \lim_{N\to\infty} \sum_{n=1}^{N} f\left(1+n\cdot\tfrac{2}{N}\right)\left(\tfrac{2}{N}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} 2\left(1+n\cdot\tfrac{2}{N}\right)\left(\tfrac{2}{N}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\tfrac{4}{N} + \tfrac{8}{N^2}n\right) =$$

$$= \lim_{N\to\infty} \left(\tfrac{4}{N}(N) + \tfrac{8}{N^2}\cdot\tfrac{N(N+1)}{2}\right) = \lim_{N\to\infty} \left(4 + 4\cdot\tfrac{N^2+N}{N^2}\right) = 4 + \lim_{N\to\infty} \left(4+\tfrac{4}{N}\right)$$

$$= 4+4+0 = 8. \qquad \text{(This is what we got in §4.1).}$$

③ $f(x)=x^2$ on $[0,2]$

Again $\Delta x = \frac{2-0}{N} = \frac{2}{N}$ and now $x_n = a+n\Delta x = 0+n\cdot\frac{2}{N} = n\cdot\frac{2}{N}$. So,

$$A = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\Delta x = \lim_{N\to\infty} \sum_{n=1}^{N} \left(n\cdot\tfrac{2}{N}\right)^2\cdot\left(\tfrac{2}{N}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \tfrac{8}{N^3}n^2 = \lim_{N\to\infty} \tfrac{8}{N^3}\cdot\tfrac{N(N+1)(2N+1)}{6}$$

$$= \lim_{N\to\infty} \frac{8\left(1+\frac{1}{N}\right)\left(2+\frac{1}{N}\right)}{6} = \frac{8(1+0)(2+0)}{6} = \frac{16}{6} = \frac{8}{3} = 2.66\ldots$$

(In §4.1, we estimated 3.75, so clearly $N=4$ was not a good enough estimate).