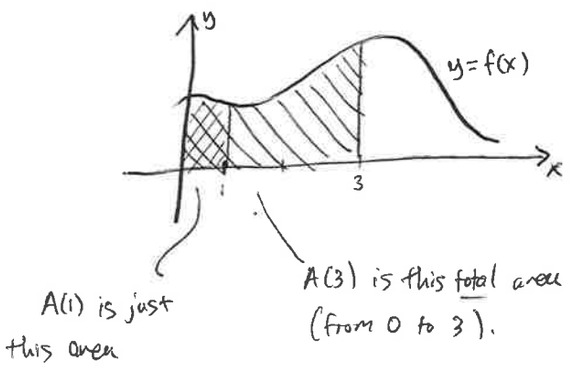
§4.3

MAIN CONCEPTS

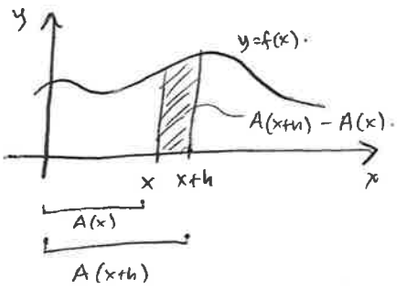
1. Fundamental Theorem of Calculus (I)
2. Signs and Interval Additivity
3. Linearity.

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1. The process in §4.2 is tedious. We'd love to compute areas more simply if possible. The trick is to consider area as a function, rather than just a number. To that end, suppose we have a function $f(x)$ and define the area function $A(x)$ to be the area under $y=f(x)$ between 0 and $x$.




Then, we can ask about the derivative of $A(x)$. By definition, this is $\lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$. First, consider the numerator $A(x+h)-A(x)$. This is the area above $[0, x+h]$ minus the area above $[0,x]$. So, it's just the area above $[x, x+h]$:




When $h$ is small, this is a rectangle with base $h$ and height $f(x)$. So for $h$ small, $A(x+h)-A(x) \approx h \cdot f(x)$. Then $A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = \lim_{h\to 0} \frac{h\cdot f(x)}{h} = f(x)$.

In other words, if we calculate area and take the derivative, we get back the original function! So "area" = "antiderivative" in some sense. To match this result, let's introduce some notation:

[NOTATION: The area under $y=f(x)$ from $x=a$ to $x=b$ is $\int_a^b f(x)\,dx$.

With this new notation, our function $A(x)$ can be defined as $A(x) = \int_0^x f(t)\,dt$,

and the result of the above discussion is:

**FUNDAMENTAL THEOREM OF CALCULUS (PART I):** If $f(x)$ is continuous on $[a,b]$, then for any $x \in (a,b)$:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

**EXAMPLES:** Compute the following derivatives.

① $\frac{d}{dx}\int_0^x t^2\,dt$.

This is exactly (FTC II) above: $\frac{d}{dx}\int_0^x t^2\,dt = x^2$.

② $\frac{d}{dx}\int_8^x \sin(t^2+1)\,dt$

Ditto: $\frac{d}{dx}\int_8^x \sin(t^2+1)\,dt = \sin(x^2+1)$ [Notice the "8" = a doesn't matter].

③ $\frac{d}{dx}\int_0^{x^2} \cos(t)\,dt$.

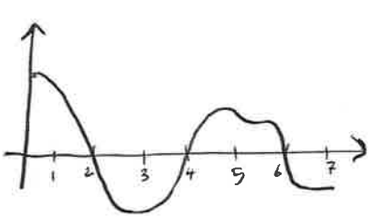
At first, we're not sure how to do this because the top bound isn't $x$. But we can write $A(x) = \int_0^x \cos(t)\,dt$ and (FTC 1) applies to it: $A'(x) = \cos(x)$. How does this help us? We use the chain rule:

$$\frac{d}{dx}\int_0^{x^2} \cos(t)\,dt = \frac{d}{dx}A(x^2) = A'(x^2)\cdot 2x = 2x\cos(x^2).$$

2 We now need to back up and look at our definitions for some specific cases. First, what happens if $f(x) < 0$? We've been talking about area <u>below</u> a curve, but now that area is underneath the x-axis. To address this, we consider this as <u>negative area</u>; then (FTC 1) works in these cases as well.

Second, we've considered $\int_a^b f(x)\,dx$ when $a < b$. What about $a = b$ or $a > b$? For $a = b$, we get a rectangle of width zero, so $\int_a^a f(x)\,dx = 0$. For $a > b$, we get a reverse orientation so we again get negative area: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

**EXAMPLES:** For the graph of $y = f(x)$ shown, determine the sign of various quantities.




① $\int_0^2 f(x)\,dx$

This is the area ~~above~~ $[0,2]$, which is <u>positive</u>.

② $\int_2^4 f(x)\,dx$

This area is <u>below</u> the x-axis, so is <u>negative</u>.

③ $\int_6^4 f(x)\,dx$

We know $\int_4^6 f(x)\,dx$ is positive (above the x-axis), so $\int_6^4 f(x)\,dx = -\int_4^6 f(x)\,dx$ is <u>negative</u>.

With these definitions, we can also talk about adding intervals. If we get the area above $[1,7]$ and add the area above $[7,8]$, we should get the area above $[1,8]$.

[INTERVAL ADDITIVITY: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

This makes sense if $a<b<c$, but even works if $b=c$ or $a>b$ or $a>c$, etc.

3 Finally, the connection between areas and antiderivatives means we inherit the usual properties:

[COEFFICIENT and SUM RULES: $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ ~~and $\int_a^b (f(x)+g(x))\,dx$~~

and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

EXAMPLES: Compute these derivatives.

① $\frac{d}{dx}\int_0^{x^2} (t+2)\,dt$

We already did a problem like this and by the Chain Rule we get

$(2x)\cdot(x^2+2)$ [Plug in $t=x^2$ and multiply by $\frac{d}{dx}(x^2)$]

② $\frac{d}{dx}\int_x^{x^2} (t^3)\,dt$.

We use interval additivity first:

$$\frac{d}{dx}\int_x^{x^2} t^3\,dt = \frac{d}{dx}\left[\int_x^0 t^3\,dt + \int_0^{x^2} t^3\,dt\right] = \frac{d}{dx}\left[-\int_0^x t^3\,dt + \int_0^{x^2} t^3\,dt\right] =$$

$$= -\frac{d}{dx}\int_0^x t^3\,dt + \frac{d}{dx}\int_0^{x^2} t^3\,dt = -(x)^3 + (x^2)^3\cdot(2x) = -x^3 + 2x^7$$

③ $\frac{d}{dx}\int_2^{\sin(x)} t\,dt$.

Again, just FTC I and Chain Rule:

$$\frac{d}{dx}\int_2^{\sin(x)} t\,dt = (\sin(x))\cdot(\cos(x)) = \sin(x)\cos(x).$$