§4.4

MAIN CONCEPTS

1. Fundamental Theorem of Calculus (II).
2. u-substitution (bounds).

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1. The connection between area and antiderivatives suggests that we should be able to compute the former using the latter. Indeed, we've found that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. But let's apply this to $f'(x)$ itself. Then $\frac{d}{dx}\int_a^x f'(t)\,dt = f'(x)$. Here we have two functions ($\int_a^x f'(t)\,dt$ and $f(x)$) and their derivatives are equal. So, by the end of §3.6, we have $\int_a^x f'(t)\,dt = f(x) + C$ for some C. But at $x=a$ we have $0 = \int_a^a f'(t)\,dt = f(a) + C$ so $C = -f(a)$ and $\int_a^x f'(t)\,dt = f(x) - f(a)$. Plugging in $x=b$ let's us solve for areas:

FUNDAMENTAL THEOREM OF CALCULUS (PART II): If $f(x)$ is differentiable, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Since $f(x)$ is an antiderivative of $f'(x)$, we've established a two-way connection between areas and antiderivatives. To further emphasize this connection, we will also refer to $\int f(x)\,dx$ as an indefinite integral and $\int_a^b f(x)\,dx$ as a definite integral.

EXAMPLES: Find the area under $y=f(x)$ above the indicated interval.

(1) $f(x) = 7$ on $[2,4]$.

We now see that we need to find an antiderivative and plug in the bounds:

$$\text{Area} = \int_2^4 f(x)\,dx = \int_2^4 7\,dx = 7x\Big|_2^4 = 7(4) - 7(2) = 28 - 14 = 14.$$

(2) $f(x) = 2x$ on $[1,3]$

As above, $\text{Area} = \int_1^3 2x\,dx = x^2\Big|_1^3 = (3)^2 - (1)^2 = 9 - 1 = 8.$

(3) $f(x) = x^2$ on $[0,2]$.

Now, $A = \int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3}$

These were the same examples in §4.1 and §4.2, but notice how much shorter our solutions have become using (FTC 2).

(4) $f(x) = 4x - 6x^5$ on $[0,1]$

Same idea: $A = \int_0^1 (4x - 6x^5)\,dx = 2x^2 - x^6\Big|_0^1 = [2(1)^2 - (1)^6] - [2(0)^2 - (0)^6]$

$= 2 - 1 - 0 + 0 = 1.$

⑤ $f(x) = 2x(x^2-1)^{90}$ on $[0,1]$.

We'll need a u-substitution so put $u = x^2-1$ and $du = 2x\,dx$. Then

$$A = \int_0^1 2x(x^2-1)^{90}\,dx = \int u^{90}\,du = \frac{1}{91}u^{91} = \frac{1}{91}(x^2-1)^{91}\Big|_0^1$$

$$= \frac{1}{91}\left((1)^2-1\right)^{91} - \frac{1}{91}\left((0)^2-1\right)^{91} = \frac{1}{91}(0) - \frac{1}{91}(-1) = \frac{1}{91}$$

② Notice in this last example, we needed a u-substitution to find the antiderivative, and put everything back in terms of x before plugging in $x=1$ and $x=0$. To save a bit of time, it would be nice if we could plug in for u directly. For this, we need to convert our bounds.

For example, in ⑤ above we had $u = x^2-1$. So our bounds convert:

$x = 0 \to u = (0)^2-1 = -1$ and $x = 1 \to u = (1)^2-1 = 0$. So, we can write $\int_0^1 2x(x^2-1)^{90}\,dx = \int_{-1}^0 u^{90}\,du$ and then evaluate without switching back to x: $\cdots = \frac{1}{91}u^{91}\Big|_{-1}^0 = \frac{1}{91}(0)^{91} - \frac{1}{91}(-1)^{91} = \frac{1}{91}$ as before.

EXAMPLES: Find the following integrals.

① $\int_0^1 3x^2\sqrt{x^3+1}\,dx$

Use $u = x^3+1$ so $du = 3x^2\,dx$ and our bounds are $x=1 \to u = (1)^3+1 = 2$, $x=0 \to u = (0)^3+1 = 1$. So

$$\int_0^1 3x^2\sqrt{x^3+1}\,dx = \int_1^2 \sqrt{u}\,du = \int_1^2 u^{1/2}\,du = \frac{u^{3/2}}{3/2}\Big|_1^2 = \frac{2}{3}\left((2)^{3/2} - (1)^{3/2}\right) = \frac{2}{3}\left(\sqrt{8}-1\right)$$

② $\int_0^2 \cos(\pi x)\,dx$.

Take $u = \pi x$, so $du = \pi\,dx$, and $x=2 \to u = \pi(2) = 2\pi$, $x=0 \to u = \pi(0) = 0$. Then

$$\int_0^2 \cos(\pi x)\,dx = \int_0^{2\pi} \cos(u)\,\frac{du}{\pi} = \frac{1}{\pi}\sin(u)\Big|_0^{2\pi} = \frac{1}{\pi}\left(\sin(2\pi) - \sin(0)\right) = 0.$$