§4.5

MAIN CONCEPTS

1. Average of a function
2. Mean Value Theorem
3. Symmetry.
4. Integrability.

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1 How do we compute averages? For finite collections, we understand this already — let's say we want the average height of students in class, and there are 60 students with heights $y_1, y_2, y_3, \dots, y_{60}$. Then the average is $\frac{y_1+y_2+\cdots+y_{60}}{60}$.

What about a more continuous variable? Let's say we go on a drive and $f(x)$ is our elevation (above sea level) when we've $x$ miles into the trip. We can ask a similar question — what's our average elevation along the trip?

We'll use the strategy we used for integrals in general — divide up the trip into $N$ segments, ask for the elevation in each segment, and take their average. So, if we start at $x=a$ and end at $x=b$, then we get $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$ as before, so average $\approx \frac{f(x_1)+f(x_2)+\cdots+f(x_N)}{N} = \frac{1}{N}\sum_{n=1}^{N} f(x_n)$. This almost looks like an integral! Let's multiply by $\frac{\Delta x}{\Delta x}$ to get it:

$$\text{average} \approx \frac{\Delta x}{\Delta x}\cdot\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \frac{1}{N\Delta x}\sum_{n=1}^{N} f(x_n)\Delta x = \frac{1}{N\cdot\frac{b-a}{N}}\sum_{n=1}^{N} f(x_n)\Delta x$$

$$= \frac{1}{b-a}\sum_{n=1}^{N} f(x_n)\Delta x.$$

Taking $N\to\infty$ turns the estimate exact:

[redacted]

**AVERAGE VALUE**: The average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$

EXAMPLES: Compute the average of $f(x)$ on the indicated interval.

① ~~$f(x)=3x^2$~~ on $[0,4]$.

By definition, the average is $\frac{1}{4-1}\int_1^4 3x^2\,dx = \frac{1}{3}\cdot\left(x^3\Big|_1^4\right) = \frac{1}{3}(4^3-1^3) = \frac{63}{3} = 21$.

② $f(x)=\sin(x)$ on $[0,\pi]$.

Ditto: average $= \frac{1}{\pi-0}\int_0^\pi \sin(x)\,dx = \frac{1}{\pi}\left[-\cos(x)\Big|_0^\pi\right] = \frac{1}{\pi}\left[-\cos(\pi)+\cos(0)\right] = \frac{2}{\pi}$.

③ $f(x) = \begin{cases} x & \text{if } x<1 \\ 2 & \text{if } x\geq 1 \end{cases}$ on $[0,2]$.

We split up the interval to match the piecewise definition.

$$\text{average} = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx\right] = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 2\,dx\right] =$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\Big|_0^1 + 2x\Big|_1^2\right] = \frac{1}{2}\left[\left(\frac{1^2}{2} - \frac{0^2}{2}\right) + (2(2) - 2(1))\right] = \frac{1}{2}\left[\frac{1}{2} + 2\right] = \frac{5}{4}.$$

2 Follow-up question: is the average value an actual value in your dataset? For finitely many, not usually — nobody is exactly average height — but for continuous data, we expect so — at some point, you're exactly at the average elevation, even if only for an instant. This observation is the Mean Value Theorem (for integrals):

MEAN VALUE THEOREM (FOR INTEGRALS): If $f(x)$ is continuous on $[a,b]$ then $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ for some $a < c < b$.

EXAMPLES: Convince yourself that MVT works for each example above, or explain why it doesn't.

① For $f(x) = 3x^2$ on $[1,4]$, we found average $= 21$. For MVT to work, we need $f(c) = 21 \longrightarrow 3c^2 = 21 \longrightarrow c^2 = 7 \longrightarrow c = \pm\sqrt{7}$. But $-\sqrt{7}$ isn't in the domain so only $c = \sqrt{7}$ works (and $1 < \sqrt{7} < 4$ as claimed).

② Now we want $\sin(c) = \frac{2}{\pi}$. Since $0 < \frac{2}{\pi} < 1$, $0 < \arcsin\left(\frac{2}{\pi}\right) < \frac{\pi}{2}$ so $c = \arcsin\left(\frac{2}{\pi}\right)$ works $\left[\sin\left(\arcsin\left(\frac{2}{\pi}\right)\right) = \frac{2}{\pi}\right]$.

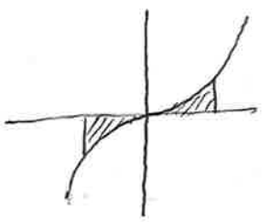
③ We want $f(c) = \frac{5}{4}$. If $c \geq 1$, then $f(c) = 2 \neq \frac{5}{4}$. But if $c < 1$, then $f(c) = c \neq \frac{5}{4}$ (since $\frac{5}{4} \not< 1$). So MVT doesn't work here. This is okay — $f(x)$ is not continuous, so MVT doesn't apply.

3 We finish the chapter with some observations. The first is about symmetry.

EXAMPLE: Find $\int_{-1}^1 x^3\,dx$.

By (FTC 2), $\int_{-1}^1 x^3\,dx = \frac{1}{4}x^4\Big|_{-1}^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(-1)^4 = \frac{1}{4} - \frac{1}{4} = 0$,

Can we understand this geometrically? Sure: $y = x^3$ is an odd function, so the negative area from $[-1,0]$ exactly cancels the positive area from $[0,1]$. In general, we have



ODD SYMMETRY: If $f(x)$ is odd $[f(-x) = -f(x)]$ then $\int_{-a}^a f(x)\,dx = 0$

Similarly, if $f(x)$ is even, it will be symmetric about the $y$-axis, so the areas

will add (instead of cancelling).

**EVEN SYMMETRY**: If $f(x)$ is even $[f(x) = f(-x)]$ then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

**EXAMPLES**: Use symmetry to simplify before evaluating.

① $\int_{-\pi}^{\pi} (x^5 + 3x^3 - 2x)\,dx$.

$f(x) = x^5 + 3x^3 - 2x$ is odd, so $\int_{-\pi}^{\pi} f(x)\,dx = 0$.

② $\int_{-\pi/2}^{\pi/2} \cos(x)\,dx$

$\cos(x)$ is even, so $\int_{-\pi/2}^{\pi/2} \cos(x)\,dx = 2\int_{0}^{\pi/2} \cos(x)\,dx = 2\left(\sin(x)\Big|_0^{\pi/2}\right) = 2\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) = 2$

③ ~~$\int_0^2 (x^3 - 3x^2 + 3x - 1)\,dx$~~ $\int_0^2 (x^3 - 3x^2 + 3x - 1)\,dx$

The interval isn't centered at 0, so symmetry doesn't seem to apply. But if we take a u-sub with $u = x - 1$, $du = dx$, $\begin{matrix} x=2 \to u = 2-1 = 1 \\ x=0 \to u = 0-1 = -1 \end{matrix}$ we get:

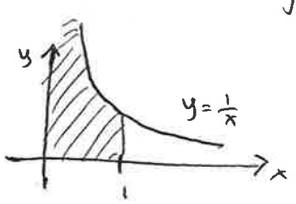
~~$\int_0^2 (x^3 - 3x^2 + 3x - 1)\,dx = \int_{-1}^{1} \left[(u+1)^3 - 3(u+1)^2 + 3(u+1) - 1\right] du$~~

$$\int_0^2 (x^3 - 3x^2 + 3x - 1)\,dx = \int_{-1}^{1} \left[(u+1)^3 - 3(u+1)^2 + 3(u+1) - 1\right] du =$$

$$= \int_{-1}^{1} \left[(u^3 + 3u^2 + 3u + 1) - (3u^2 + 6u + 3) + (3u + 3) - 1\right] du =$$

$$= \int_{-1}^{1} u^3\,du = 0 \qquad \text{since } u^3 \text{ is odd.}$$

④ Finally, a technical note. We've been writing $\int_a^b f(x)\,dx$, but this is defined by a limit, so it might not exist sometimes. E.g. $\int_0^1 \frac{1}{x}\,dx$ computes a potentially infinite area. To handle these, we have the following result.




**THEOREM**: If $f(x)$ is bounded on $[a,b]$ and continuous except at finitely many points, then $\int_a^b f(x)\,dx$ exists.

**DEFINITION**: When $\int_a^b f(x)\,dx$ exists, we say $f(x)$ is integrable on $[a,b]$.

**EXAMPLE**: $f(x) = \frac{1}{x}$ is integrable on $[1,3]$, but isn't necessarily on $[-1,1]$.