§5.1

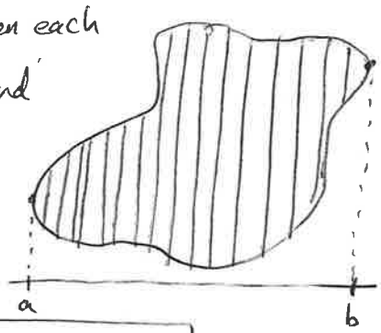
Main Concepts

1. Areas (again)
2. Horizontal Slicing

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1. Chapter §4 established well the connection between antiderivatives and integrals, and using this to find areas. But we were restricted to the area under a curve and above the x-axis. Our work now is to generalize this to any region in the plane.

As before, we start by making vertical cuts. Then each cut is approximately a rectangle with width Δx and height $y_{top} - y_{bottom}$. So, the area estimate is $\sum_{n=1}^{N} (top - bottom)\,\Delta x$ and by taking $N \to \infty$, we get



$$\boxed{A = \int_a^b (top - bottom)\,dx = \int_a^b (y_2 - y_1)\,dx.}$$

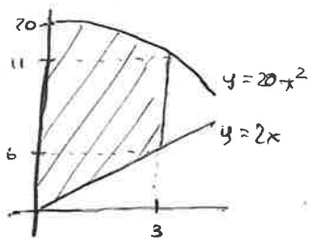
Examples: Find the area described.

① Above $y = x$ and below $y = x^2$, for $2 \le x \le 8$.

We're given "top, bottom, a, b" — so just plug it in: $A = \int_2^8 ((x^2) - (x))\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_2^8$

$= \frac{1}{3}(8^3 - 2^3) - \frac{1}{2}(8^2 - 2^2) = \frac{504}{3} - \frac{60}{2} = 138.$

② Between $y = 20 - x^2$ and $y = 2x$, for $0 \le x \le 3$



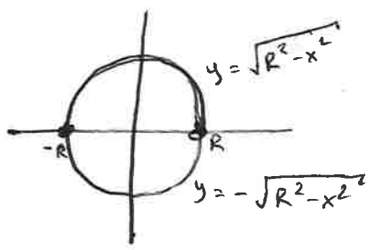
We're given all the info, but which is "top" and which is "bottom"? Graphing can help us, and then it's clear that $20 - x^2 \ge 2x$ on the interval $[0,3]$. So, $A = \int_0^3 ((20 - x^2) - (2x))\,dx = 20x - \frac{1}{3}x^3 - x^2 \Big|_0^3$

$= 20(3-0) - \frac{1}{3}(3^3 - 0^3) - (3^2 - 0^2) = 60 - 9 - 9 = 42.$

③ Inside the circle $x^2 + y^2 = R^2$ of radius R.

We already know the answer — πR^2 — but let's use integration.

We need top & bottom, so solve for y: $y^2 = R^2 - x^2 \to y = \pm\sqrt{R^2 - x^2}$



The bounds are $-R \le x \le R$ (from the picture), so

$$A = \int_{-R}^{R} \left(\left(\sqrt{R^2 - x^2}\right) - \left(-\sqrt{R^2 - x^2}\right)\right) dx = \int_{-R}^{R} 2\sqrt{R^2 - x^2}\,dx = 4\int_0^R \sqrt{R^2 - x^2}\,dx$$

where we used symmetry (even function) for the last step. To solve this, we'll use a substitution: $u = \arcsin(x/R)$ $\quad du = \frac{1}{\sqrt{1 - x^2/R^2}} \cdot \frac{1}{R}\,dx = \frac{1}{\sqrt{R^2 - x^2}}\,dx$

$x = 0 \to u = \arcsin(0) = 0$

$x = R \to u = \arcsin(1) = \pi/2.$

So, $A = 4\int_0^{\pi/2} \sqrt{R^2 - x^2} \cdot \left(\sqrt{R^2 - x^2}\,du\right) = 4\int_0^{\pi/2} (R^2 - x^2)\,du = 4\int_0^{\pi/2} (R^2 - (R\sin(u))^2)\,du =$

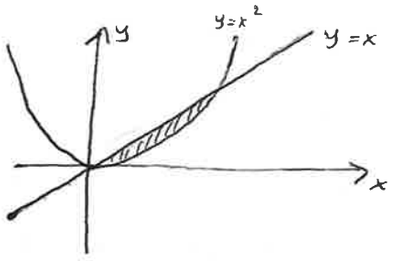
$= 4\int_0^{\pi/2}(R^2 - R^2\sin^2(u))du = 4R^2\int_0^{\pi/2}(1-\sin^2(u))du = 4R^2\int_0^{\pi/2}\cos^2(u)du.$

To finish the problem, we'll use the trig identity $\cos^2(u) = \frac{1+\cos(2u)}{2}$. Then

$A = 4R^2\int_0^{\pi/2}\frac{1+\cos(2u)}{2}du = 2R^2\left(u + \frac{1}{2}\sin(2u)\Big|_0^{\pi/2}\right) = 2R^2\left[\left(\frac{\pi}{2}-0\right) + \frac{1}{2}\left(\sin(\pi)-\sin(0)\right)\right]$

$= 2R^2\left(\frac{\pi}{2} - 0 + 0 - 0\right) = \pi R^2$, as we hoped for!

④ Bounded by $y = x$ and $y = x^2$.

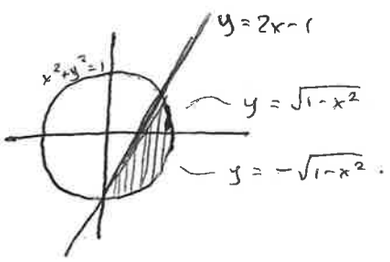


Unlike the previous problems, there's no "a" or "b". A picture makes it clear why: we can find the bounds from the setup as the intersection points. So, if

$y = x$ and $y = x^2$ then $x = x^2 \longrightarrow x^2 - x = 0 \longrightarrow x(x-1) = 0$

so $x = 0$ and $x = 1$ are the bounds. We also see $x \geq x^2$ in this region, so $A = \int_0^1 (x - x^2)dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{2}(1^2-0^2) - \frac{1}{3}(1^3-0^3) = \frac{1}{6}$.

⑤ Inside the circle $x^2+y^2=1$ and right of the line $y = 2x-1$.

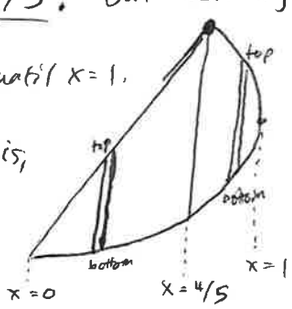


Again, there are no bounds, so let's find them:

$x^2 + (2x-1)^2 = 1 \longrightarrow x^2 + (4x^2 - 4x + 1) = 1$

$\longrightarrow 5x^2 - 4x = 0 \longrightarrow x(5x-4) = 0$

$\longrightarrow \underline{x = 0}$ or $5x - 4 = 0 \longrightarrow 5x = 4 \longrightarrow \underline{x = 4/5}$. But actually, we see that the region doesn't stop at $x = 4/5$, it continues until $x = 1$.

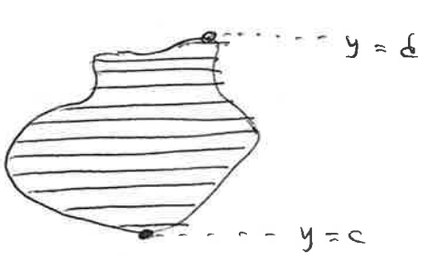


Second, what is "top" and "bottom" here? The problem is, it changes — before $x = 4/5$ it goes circle → line; but after $x = 4/5$ it goes circle → circle. So, we add up the two parts separately:

$$A = \int_0^{4/5}\left((2x-1) - \left(-\sqrt{1-x^2}\right)\right)dx + \int_{4/5}^1\left(\left(\sqrt{1-x^2}\right) - \left(-\sqrt{1-x^2}\right)\right)dx.$$

and for brevity I won't solve this now.

[2] Notice that for ⑤ above, we had to split up the region. Sometimes this is unavoidable, but sometimes we can get around it by "turning our head". That is, there is nothing sacred about "x" so we can instead try slicing horizontally. Then by similar reasoning: $A = \int_c^d (\text{right} - \text{left})dy = \int_c^d (x_2 - x_1)dy$.



EXAMPLE: Redo ⑤ above with horizontal slicing.

We immediately see one advantage: "right" = "circle" and "left" = "line" the entire time, so we don't need to split things up. This time, we need to solve for x_2 and x_1, so we get $x_2 = \text{"right"} = \sqrt{1-y^2}$ and $x_1 = \text{"left"} = \frac{y+1}{2}$ by solving for x in

$x^2+y^2=1$ and $y = 2x-1$.

Then we need the bounds $y=c$ & $y=d$. We know $x=0$ & $x=4/5$, so plugging into the equation of the line gives $c = 2(0)-1 = -1$ and $d = 2(\frac{4}{5})-1 = \frac{3}{5}$. So in all: $A = \int_{-1}^{3/5} \left(\left(\sqrt{1-y^2}\right) - \left(\frac{y+1}{2}\right)\right) dy$.

[Again, I won't solve this, but you can test both of these integrals on online tools for integrals — you'll get the same answer $\left(\frac{\pi}{2} - \arctan\left(\frac{1}{2}\right) - \frac{2}{5}\right)$.]