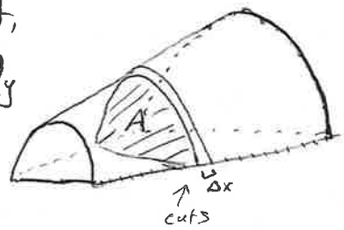
§5.2

MAIN CONCEPTS

1. Volumes and Cross-sections
2. Solids of Revolution [Disks/Washers]

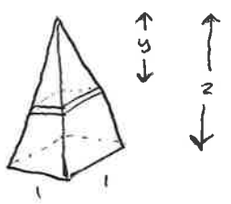
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1. Let's jump up one dimension. Instead of calculating areas, let's see how the integral can be used to compute volumes. Given a solid object, let's "cut it" into thin slices. Then each slice is approximately a prism — the volume is the cross-sectional area times the thickness. I.e. Volume of Slab $\approx \Delta x \cdot A$. So, the total volume is $\approx \sum_{n=1}^{N} A \cdot \Delta x$ and taking $N \to \infty$ gives $V = \int_a^b A\,dx$.



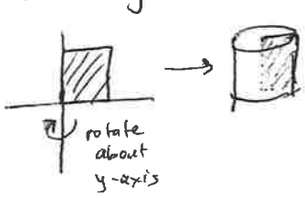
EXAMPLE: Find the volume of a pyramid with square base of side length 1 and height 2.

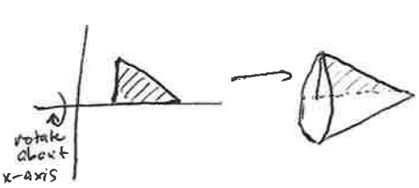
We'll make horizontal cuts at a height y from the top and thickness $\Delta y \rightsquigarrow dy$. The cross-section is a square of side length x so $A = x^2$. Then $V = \int_0^2 x^2\,dy$. To evaluate this, we need x in terms of y. By similar triangles: $\frac{x}{1} = \frac{y}{2}$ so $x = y/2$. So

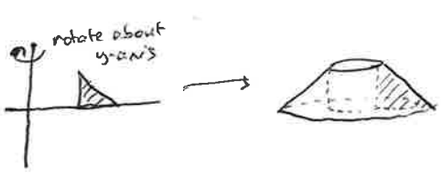


$$V = \int_0^2 \left(\frac{y}{2}\right)^2 dy = \frac{1}{4}\int_0^2 y^2\,dy = \frac{1}{12}y^3\Big|_0^2 = \frac{8}{12} - \frac{0}{12} = \frac{2}{3}.$$

2. The most common case we'll encounter in this class of solids where we can understand the cross-sectional area are called <u>solids of revolution</u>. These are what you get if you take a region in the xy-plane, and rotate it around some axis in 3d-space. E.g.







Our technique to find the volume is the same as above: slice and add. But now we have very special shapes, so things can be simplified.

Let's say we cut the original region perpendicular to the axis of rotation:



Rotating just that slice gives a "washer" whose volume is ~~Area~~ (Area)(thickness), and Area $= \pi R^2 - \pi r^2$ (big circle minus little circle).

This gives a general strategy:

Method of Washers:

① Slice your region perpendicular to the axis of rotation.

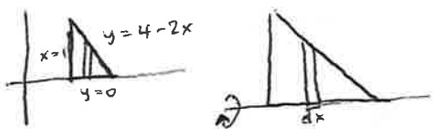
② Identify thickness (usually dx or dy), R, and r.

③ Then $V = \int_a^b (\pi R^2 - \pi r^2)(\text{thickness})$.

④ Rewrite everything (including bounds) in terms of the thickness variable, and evaluate the integral.

EXAMPLES: Find the volume when the specified region is rotated around the indicated line.

① The triangle bounded by $x=1$, $y=0$, and $y=4-2x$ rotated about x-axis.



Our process tells us first to take a vertical slice. Then thickness $= dx$, $r = 0$ (since the slice is touching the axis of rotation), and $R = y = 4-2x$. So,

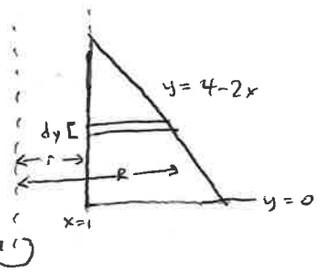
$$V = \int_a^b (\pi R^2 - \pi r^2)\,dx = \int_a^b \pi (4-2x)^2\,dx.$$

(We used $R = 4-2x$ since it had dx instead of $R = y$)

Finally, we find the bounds. $a = 1$ since we start at $x=1$, and the ending bound is where $y = 4-2x$ and $y=0$ intersect, so $4-2x=0 \to 2x=4 \to x=2$.

I.e. $V = \int_1^2 \pi(4-2x)^2\,dx = \pi\left(16x - 8x^2 + \frac{4}{3}x^3\right)\Big|_1^2 = \frac{4}{3}\pi$.

② The same region, rotated about the y-axis.



Now our slices go the other way. Then thickness $= dy$, $r = 1$, and $R = x$ from the picture, so $V = \int_a^b (\pi x^2 - \pi \cdot 1^2)\,dy$.

Now we want everything in terms of y. From $y = 4-2x$ we get $x = 2 - \frac{1}{2}y$. Our lower bound is $y=0$, and our upper is the intersection of $x=1$ with $y=4-2x$, so $y = 4-2(1) = 4-2 = 2$. I.e. $V = \int_0^2 (\pi(2-\frac{1}{2}y)^2 - \pi)\,dy$ which I'll leave you to solve.