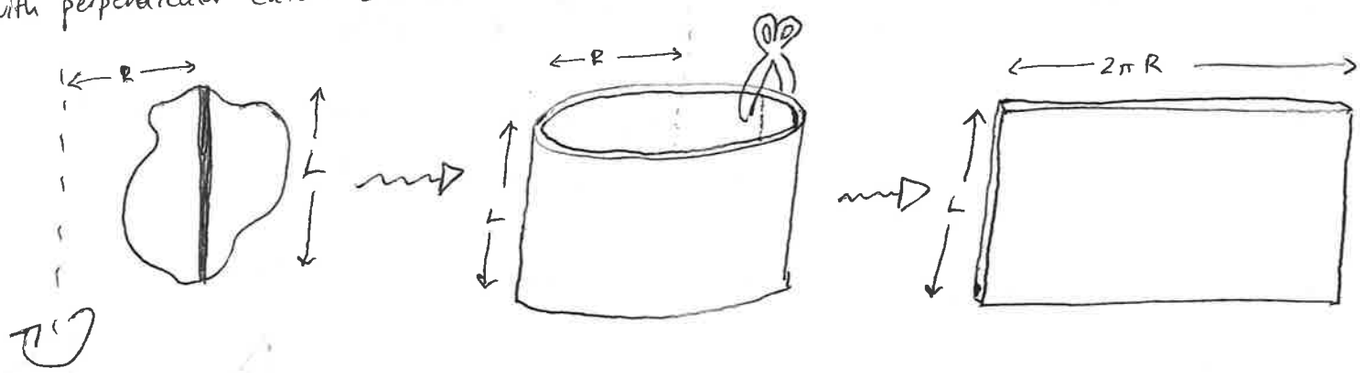
§5.3

MAIN CONCEPT

1 Solids of Revolution [shells]

1 In §5.2, we saw one way to compute the volume of a solid of revolution. But we can ask — why start with perpendicular cuts? Indeed, we don't need to, and get another strategy by not doing so.



We'll start our analysis the same way: take a slice (but now parallel to the axis) and rotate it about the axis. The result is a cylindrical shell — an incredibly thin sheet in the shape of a cylinder. But now we go one further — to compute its volume, imagine cutting along the length and unwrapping it. Then, we have

Volume of shell = (Length)(width)(thickness) = (Length)(Circumference)(thickness)

$= (L)(2\pi R)(\text{thickness})$.

So $V = \int_a^b (2\pi R)(L)(\text{thickness})$. In general:

METHOD OF SHELLS: ① Slice your region parallel to the axis of rotation.

② Identify thickness (= dx or dy), R, and L.

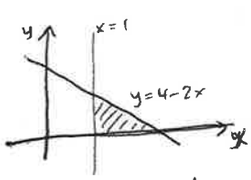
③ Then $V = \int_a^b 2\pi R L (\text{thickness})$.

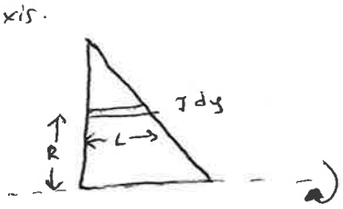
④ Switch everything to thickness variable and evaluate.

EXAMPLES: Find the indicated volumes using shells.

① The region bounded by $x=1$, $y=0$, $y=4-2x$ about the x-axis.

We sketch the region, then make the corresponding parallel cut.



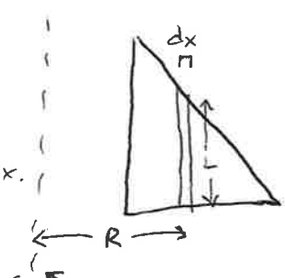


thickness = dy, $R = y$, and $L = x_2 - x_1 = \left(\frac{4-y}{2}\right) - (1) = 1 - \frac{1}{2}y$. The bounds are $y=0$ and $y = 4-2(1) = 2$.

So $V = \int_0^2 2\pi(y)\left(1-\frac{1}{2}y\right)dy = \pi\int_0^2 (2y - y^2)dy = \pi\left(y^2 - \frac{1}{3}y^3\Big|_0^2\right) = \pi\left(4 - \frac{8}{3}\right) = \frac{4}{3}\pi$

(compare to previous section).

② Same region about the y-axis.



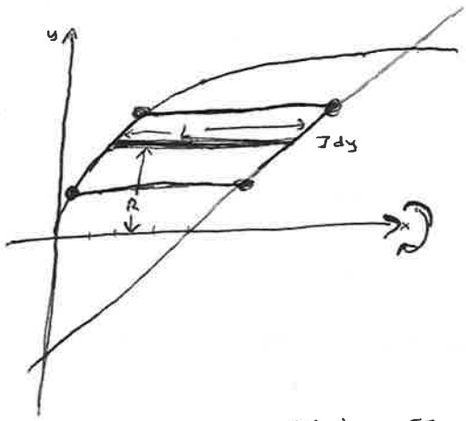
Now thickness = dx, $R = x$, and $L = y_2 - y_1$

$= (4-2x) - (0) = 4-2x$.

The bounds are $x=1$ and the intersection of $y = 4-2x$ with $y=0 \longrightarrow x = 2$.

So, $V = \int_1^2 2\pi x(4-2x)\,dx = 2\pi \int_1^2 (4x-2x^2)\,dx = 2\pi\left(2x^2 - \frac{2}{3}x^3\Big|_1^2\right) = 2\pi\left(6 - \frac{14}{3}\right) = \frac{8}{3}\pi$

③ The region bounded by $y=\sqrt{x}$, $y=1$, $y=2$, and $y=x-4$ about the x-axis.



Now thickness $= dy$, $R = y$, and $L = x_2 - x_1$.

Solving the line equation gives $x_2 = y+4$ and the curve gives $x_1 = y^2$. So $L = (y+4)-(y^2)$.

The bounds are $y=1$ to $y=2$. So

$$V = \int_1^2 2\pi(y)(y+4-y^2)\,dy$$

$$= 2\pi \int_1^2 (y^2+4y-y^3)\,dy = 2\pi\left(\frac{1}{3}y^3 + 2y^2 - \frac{1}{4}y^4\Big|_1^2\right) = 2\pi\left(\frac{7}{3} + 6 - \frac{15}{4}\right) = \frac{55}{6}\pi.$$

As an exercise, try doing ③ with washers (it's much harder!).