§5.4

MAIN CONCEPTS

1. Parameterized Curves
2. Arc Length

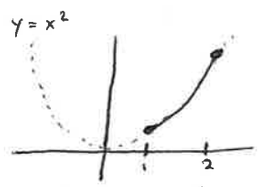
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1. We now consider another application of the integral – lengths of curves. To begin, we need a way to write curves that aren't just of the form $y = f(x)$. For this, we introduce a parameter t. You can think of t as "time" – so to specify a curve, I should tell you the x- and y-coordinates at all relevant times.

DEFINITION: A parameterized curve is a pair of functions $x = x(t)$ and $y = y(t)$, possibly combined with bounds $a \le t \le b$.

EXAMPLES: Sketch the following curves.

(1) $x(t) = t$, $y(t) = t^2$, $1 \le t \le 2$.



We can eliminate t: $y = t^2 = (x)^2 = x^2$

So this is on the parabola $y = x^2$. But $x = t$ is between 1 and 2 only.

(2) $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$.

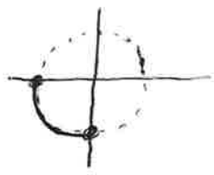
We can't eliminate t and solve for y here, but just one is enough:

$x^2 + y^2 = (\cos(t))^2 + (\sin(t))^2 = \cos^2(t) + \sin^2(t) = 1$ so $x^2 + y^2 = 1$ is a circle. We have $0 \le t \le 2\pi$, so it goes through a full period:



(3) $x(t) = \cos(t)$, $y(t) = \sin(t)$, $\pi \le t \le \frac{3\pi}{2}$.

This is still on the unit circle, but $-1 \le x \le 0$ and $-1 \le y \le 0$



2. Now, how should we find the lengths of these curves? The idea is to cut it into short pieces. Then each piece is approximately a straight line:



If we go from time t to time $t+h$, then x goes from $x(t)$ to $x(t+h)$ and y goes from $y(t)$ to $y(t+h)$. So,

Short segment length $= \sqrt{(x(t+h) - x(t))^2 + (y(t+h) - y(t))^2}$ by the distance formula.

Multiplying by $\frac{h}{h}$ gives: Short segment length $= h\sqrt{\left(\frac{x(t+h) - x(t)}{h}\right)^2 + \left(\frac{y(t+h) - y(t)}{h}\right)^2}$.

But as $h = \Delta t \to 0$, we get that these expressions are derivatives! So:

ARC LENGTH FORMULA: The length of $x(t), y(t)$, $a \le t \le b$ is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

EXAMPLES: Find the arc lengths of these curves.

① $x(t) = R\cos(t)$, $y(t) = R\sin(t)$, $0 \le t \le 2\pi$

This is a circle of radius R so we know the answer — $2\pi R$. Let's check:

$x'(t) = -R\sin(t)$, $y'(t) = R\cos(t)$, so

$$\text{Length} = \int_0^{2\pi} \sqrt{(-R\sin(t))^2 + (R\cos(t))^2}\, dt = \int_0^{2\pi} \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}\, dt = R\int_0^{2\pi} \sqrt{\sin^2(t)+\cos^2(t)}\, dt$$

$$= R\int_0^{2\pi} 1\, dt = R\,t\Big|_0^{2\pi} = 2\pi R.$$

② $x(t) = t^3$, $y(t) = 2t^2$, $0 \le t \le 1$

Now $x'(t) = 3t^2$ and $y'(t) = 4t$, so

$$\text{Length} = \int_0^1 \sqrt{(3t^2)^2 + (4t)^2}\, dt = \int_0^1 \sqrt{9t^4 + 16t^2}\, dt = \int_0^1 t\sqrt{9t^2+16}\, dt$$

To solve this, let $u = 9t^2 + 16$, so $du = 18t\,dt$. Then

$t = 1 \rightarrow u = 9\cdot 1^2 + 16 = 25$

$t = 0 \rightarrow u = 9\cdot 0^2 + 16 = 16$

$$\text{Length} = \int_{16}^{25} \frac{1}{18}\sqrt{u}\, du = \frac{1}{18}\,\frac{u^{3/2}}{3/2}\Big|_{16}^{25} = \frac{1}{27}\left(25^{3/2} - 16^{3/2}\right) = \frac{1}{27}(125 - 64) = \frac{61}{27}.$$