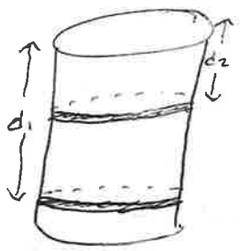
§5.5

MAIN CONCEPT

1 Energy to lift a fluid.

---

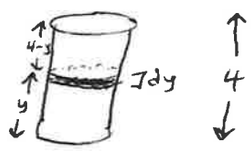
1 Finally, one more application of the integral. Suppose we want to lift some fluid up against gravity. This takes energy, and physics tells us that the energy required is weight × distance. The issue is, when pumping liquid upwards, for example, the distance is not constant: particles near the bottom of the tank have further to go than those near the top. But we know what to do: take slices! Each slice has a fixed vertical distance to travel. If ρ is the density of the fluid, then adding up the slices gives



ENERGY TO PUMP FLUID: $E = \int_a^b \rho g \,(\text{volume of slice})(\text{distance})$ where g = gravity.

EXAMPLES: Find the energy needed to empty the tanks out over the top.

① Cylindrical tank of radius 2 and height 4 with $\rho g = 3$.



We'll measure y from the base to the slice. Then

volume of slice $= \pi \cdot (2)^2 \cdot dy = 4\pi\, dy$ and

distance $= 4 - y$ so

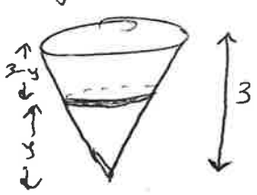
$$E = \int_0^4 (3)(4\pi\, dy)(4-y) = 12\pi \int_0^4 (4-y)\, dy = 12\pi\left(4y - \tfrac{1}{2}y^2\Big|_0^4\right) = 96\pi.$$

② Same container, but only half-full initially.

Nothing changes except the bounds:

$$E = \int_0^2 3(4\pi\, dy)(4-y) = 12\pi \int_0^2 (4-y)\, dy = 12\pi\left(4y - \tfrac{1}{2}y^2\Big|_0^2\right) = 72\pi.$$

③ Conical container (point-down) with top radius = 4 and height = 3 with $\rho g = 2$.



Now volume of slice $= \pi R^2 dy$ but R depends on y. By similar triangles: $\frac{R}{4} = \frac{y}{3}$ so $R = \frac{4}{3}y$ i.e.

volume of slice $= \frac{16}{9}\pi y^2 dy$. Then distance $= 3 - y$ so

$$E = \int_0^3 (2)\left(\tfrac{16}{9}\pi y^2 dy\right)(3-y) = \tfrac{32}{9}\pi \int_0^3 (3y^2 - y^3)\, dy = \tfrac{32}{9}\pi\left(y^3 - \tfrac{1}{4}y^4\Big|_0^3\right) = 24\pi.$$