§6.1

MAIN CONCEPTS

1. Logarithms, Before
2. Definition and Properties
3. Derivatives and Integrals with ln
4. Logarithmic Differentiation.

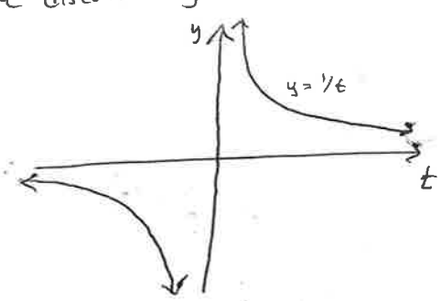
1. Prior to this class, you may have seen the logarithm, defined as an inverse to exponentiation. I.e. if $a^b = c$ then $\log_a(c) = b$. It is then common to consider the "natural log" or $\ln(x) = \log_e(x)$ function. But what makes this "natural"? What is $e = 2.718\ldots$?

Our goal for this section is to answer these questions by redefining $\ln(x)$ in a calculus way. Then, we should show that our new definition satisfies all the properties we would expect for $\log_e(x)$. Namely, since $e^0 = 1$, we want $\log_e(1) = 0$. Since $e^{a+b} = e^a \cdot e^b$, we want $\ln(ab) = \ln(a) + \ln(b)$. Since $(e^a)^b = e^{ab}$, we want $\ln(a^b) = b \cdot \ln(a)$. Finally, since e^x is positive for any x, we want $\ln(x)$ to have domain $x > 0$.

2. Let's start with the (unmotivated) definition:

DEFINITION: $\ln(x) = \int_1^x \frac{1}{t}\,dt$

Note that the function $f(t) = \frac{1}{t}$ is not defined at $t=0$, the discontinuity is not removable, and the function is not bounded near $t=0$ (it goes to $\pm\infty$), so we don't know if we can integrate on any interval containing 0. So, the above definition only makes sense for $x > 0$. But this is what we wanted! So far so good.



Let's try to check some other properties. First,

$\ln(1) = \int_1^{(1)} \frac{1}{t}\,dt = 0$ ✓ since we're integrating on an empty domain.

Next, let's consider $\ln(ax)$. We want to show it equals $\ln(a) + \ln(x)$; here's one approach:

First, by the Fundamental Theorem of Calculus, Part 1 (FTC1),

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

Then, by the chain rule,

$$\frac{d}{dx}\left(\ln(ax)\right) = \frac{1}{(ax)} \cdot (a) = \frac{1}{x}.$$

These two functions have the same derivative, so they must be equal up to a constant: $\ln(ax) = \ln(x) + C$. To determine C, let's plug in $x=1 \longrightarrow \ln(a\cdot 1) = \ln(1) + C$

$\rightarrow \ln(a) = 0 + C \longrightarrow C = \ln(a)$. So, in conclusion, $\ln(ax) = \ln(x) + \ln(a)$ as we hoped to show. In a similar vein, we can show the remaining fact: $\ln(x^k) = k\ln(x)$.

In summary:

PROPERTIES OF $\ln(x)$: • $\ln(1) = 0$ • $\ln(ab) = \ln(a) + \ln(b)$ • $\ln(a^b) = b\ln(a)$

3 Note that we can now take derivatives of functions involving the natural log, just as we did above. Let's see some examples:

EXAMPLES: Compute $f'(x)$ in each.

1. $f(x) = \ln(x)$

   As we noted above, $f'(x) = \frac{1}{x}$.

2. $f(x) = \ln(10x)$.

   We can use the chain rule: $f'(x) = \frac{1}{(10x)} \cdot (10) = \frac{1}{x}$.

   Alternatively, we can use log properties: $f(x) = \ln(10) + \ln(x)$, so

   $f'(x) = 0 + \frac{1}{x} = \frac{1}{x}$ since $\ln(10)$ is a constant.

3. $f(x) = \ln(x^3 + 1)$

   Again, the chain rule: $f'(x) = \frac{1}{x^3+1} \cdot (3x^2 + 0) = \frac{3x^2}{x^3+1}$

   Note: there isn't a log rule to help this time.

4. $f(x) = x\ln(x)$

   Now we need a product rule: $f'(x) = (1)(\ln(x)) + (x)\left(\frac{1}{x}\right) = \ln(x) + 1$.

5. $f(x) = \ln|x|$.

   There are two cases: for $x > 0$, $|x| = x$, so $f(x) = \ln(x)$ and $f'(x) = \frac{1}{x}$.

   For $x < 0$, $|x| = -x$, so $f(x) = \ln(-x)$ and $f'(x) = \frac{1}{(-x)} \cdot (-1) = \frac{1}{x}$.

   In other words, $f'(x) = \frac{1}{x}$ for all $x \neq 0$.

This last example is especially enlightening — what we found is a function with domain $(-\infty, 0) \cup (0, \infty)$ [ie. all x except $x=0$] with derivative $\frac{1}{x}$. We can therefore now take a general antiderivative:

RESULT: $\int \frac{1}{x}\,dx = \ln|x| + C$ for $x \neq 0$.

Combined with our previous toolbox for integrals, we can solve even more problems.

EXAMPLES: Evaluate each integral.

① $\int \frac{1}{x}\,dx$

This is $\ln|x|+C$ as we've just concluded.

② $\int \frac{2}{x}\,dx$

This is $= 2\int \frac{1}{x}\,dx = 2\ln|x| + C$.

③ $\int \frac{5}{x-1}\,dx$

Use the substitution $u = x-1 \rightarrow du = dx$. Then

$$\int \frac{5}{x-1}\,dx = \int \frac{5}{u}\,du = 5\ln|u| + C = 5\ln|x-1| + C.$$

④ $\int \frac{1}{3x+7}\,dx$

Same idea: $u = 3x+7 \rightarrow du = 3\,dx$, Then

$$\int \frac{1}{3x+7}\,dx = \int \frac{1}{u}\cdot\frac{du}{3} = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|3x+7| + C.$$

⑤ $\int \frac{x^2}{x^3+6}\,dx$

Same idea: $u = x^3+6 \rightarrow du = 3x^2\,dx$. Then

$$\int \frac{x^2}{x^3+6}\,dx = \int \frac{x^2}{u}\cdot\frac{du}{3x^2} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+6| + C.$$

⑥ $\int \frac{x^2+1}{x+1}\,dx$

We'd like to try the same, but it won't get rid of the x^2. So, let's start with polynomial long division. This gives $\frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1}$. Then

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+0x+1 \\ & x^2+x \\ \hline & -x+1 \\ & -x-1 \\ \hline & 2 \end{array}$$

$$\int \frac{x^2+1}{x+1}\,dx = \int \left(x - 1 + \frac{2}{x+1}\right)dx = \frac{1}{2}x^2 - x + 2\ln|x+1| + C.$$

⑦ $\int \tan(x)\,dx$

This seems unrelated at first, but recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. So, like before, let's take $u = \cos(x)$ so $du = -\sin(x)\,dx$ and

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{\sin(x)}{u}\cdot\frac{du}{-\sin(x)} = -\ln|u| + C = -\ln|\cos(x)| + C.$$

④ Finally, let's end this section with a useful technique. When taking derivatives of expressions with lots of multiplication and division, it can be useful to first take ln of both sides, since $\ln(ab) = \ln(a) + \ln(b)$ and it's easier to differentiate a sum than a product. This process is called <u>logarithmic differentiation</u>.

EXAMPLES: Find $f'(x)$ using logarithmic differentiation.

① $f(x) = \dfrac{(x+1)(x+2)}{x^2}$

We could do this with product/quotient rules, but it might get messy. Instead, take ln:

$$\ln(f(x)) = \ln\left(\frac{(x+1)(x+2)}{x^2}\right) = \ln(x+1) + \ln(x+2) - \ln(x^2) = \ln(x+1) + \ln(x+2) - 2\ln(x)$$

Now the derivative of the right-hand side is much simpler. The left requires a chain rule:

$$\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}\left[\ln(x+1) + \ln(x+2) - 2\ln(x)\right]$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x}$$

Then, multiply by $f(x)$ to solve for $f'(x)$.

$$f'(x) = f(x)\left[\frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x}\right]$$

$$\boxed{f'(x) = \frac{(x+1)(x+2)}{x^2}\left[\frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x}\right].}$$

② $f(x) = x^x$

This one we <u>can't</u> do with previous techniques. It looks like a power rule, but the exponent isn't constant, so this doesn't work. Instead, we take ln:

$$\ln(f(x)) = \ln(x^x) = x \cdot \ln(x).$$

Now we <u>can</u> take the derivative with a product rule:

$$\frac{1}{f(x)} \cdot f'(x) = (1)(\ln(x)) + (x)\left(\frac{1}{x}\right) = \ln(x) + 1$$

$$f'(x) = f(x)(\ln(x) + 1)$$

$$f'(x) = x^x(\ln(x) + 1).$$