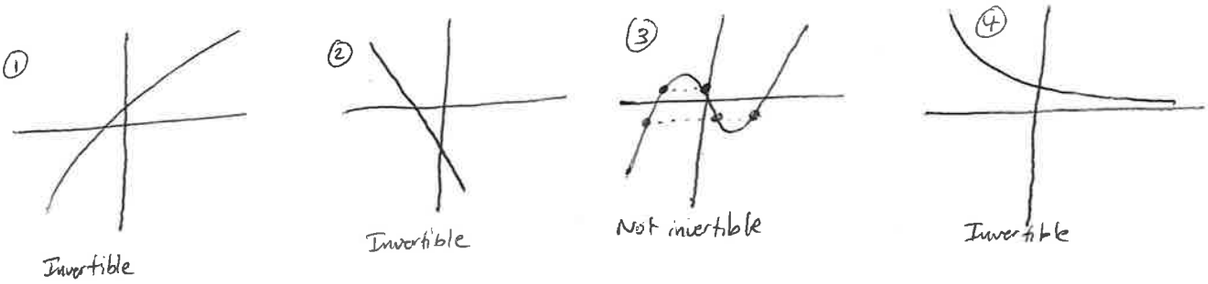
§6.2

## Main Concepts

1. Finding inverse functions
2. Inverse function theorem

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1 A function is any rule that takes an input and returns an output solely depending on that input. For some functions, we can reverse this: given the output, figure out which input produced it. Of course, this is impossible if two different inputs produce the same output. Visually, the graph of a function can tell us this: if a horizontal line intersects the graph more than once, then all of those inputs (x-values) provide the same output (y-value). So, such a function cannot be reversed: we say it is not <u>invertible</u>.




The pictures above suggest a result which is indeed true:

**Theorem**: If a function is strictly monotonic (increasing or decreasing), it is invertible.

For example, graphs ② & ④ are strictly decreasing and graph ① is strictly increasing, so they are invertible. How can we find these inverses? Since the goal is to ~~scramble~~ switch the roles of the input & output, we do exactly that — switch $x$ & $y$ and solve for $y$.

**Example**: Show that $f(x) = 2x + 6$ is invertible and find the inverse.

To show that $f(x)$ is invertible, notice $f'(x) = 2$ is <u>positive</u>. So, $y = f(x)$ is always increasing, and so $f(x)$ is invertible by the theorem above. To find the inverse write $y = f(x)$:

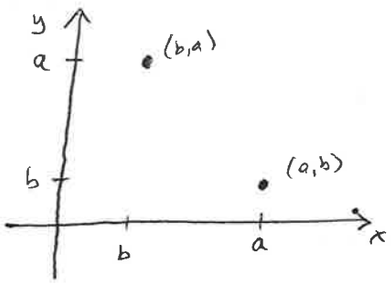
$$y = 2x + 6 \quad \text{— then switch } x \text{ & } y \longrightarrow \quad x = 2y + 6, \text{ and solve for } y:$$

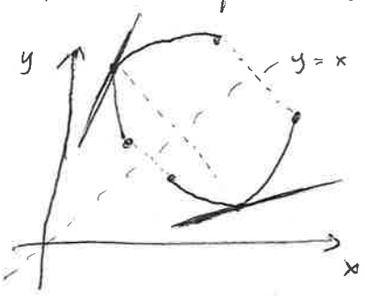
$$x - 6 = 2y \longrightarrow \boxed{y = \frac{x-6}{2}.}$$ So the inverse function is $g(x) = \frac{x-6}{2}$.

In the end, the inverse function is a function, so we gave it a name ($g(x)$), but it is often useful to give a name that refers back to the original function. We'll typically write $f^{-1}(x)$ for the inverse of $f(x)$, whenever it exists.

2 Now, let's see if we can do calculus with these inverse functions. Specifically, we can ask: what is the derivative of $f^{-1}(x)$?

One way to determine this is graphically, using the correspondence between derivatives and slope. So, we should ask, what is the graph of $y = f^{-1}(x)$? We do graphically what we did algebraically: switch $x$ & $y$. In other words, if the point $(a,b)$ is on




the graph of $y = f(x)$, then $(b,a)$ is on the graph of $y = f^{-1}(x)$. In other words, we "flip" or "reflect" the graph over the line $y = x$.

But now, we can see what this does to tangent lines: the line $y = mx + b$ "flips" to become $x = my + b \rightarrow x - b = my \rightarrow y = \frac{1}{m}x - \frac{b}{m}$. So, a slope of $m$ becomes a slope of $\frac{1}{m}$. This is the result:

**INVERSE FUNCTION THEOREM:** If $f(x)$ is differentiable at $x = a$ and $f'(a) \neq 0$, then $f^{-1}(x)$ is differentiable at $b = f(a)$, and $(f^{-1})'(b) = \frac{1}{f'(a)}$.

Notice we need to assume $f'(a) \neq 0$ since we can't divide by zero.

**EXAMPLES:** Find $(f^{-1})'(4)$ for each function $f(x)$.

① $f(x) = x^3 - 4$.

I.e. $y = x^3 - 4$, so we can find the inverse function directly:

$x = y^3 - 4 \rightarrow x + 4 = y^3 \rightarrow y = (x+4)^{1/3}$. So $f^{-1}(x) = (x+4)^{1/3}$. Then

$(f^{-1})'(x) = \frac{1}{3}(x+4)^{-2/3}$ and so $(f^{-1})'(4) = \frac{1}{3}(4+4)^{-2/3} = \frac{1}{3}(8)^{-2/3} = \frac{1}{3} \cdot \frac{1}{4} = \boxed{\frac{1}{12}}$

Alternatively, we can use the inverse function theorem. Since $f(2) = 2^3 - 4 = 8 - 4 = 4$, the point $(2,4)$ is on the graph. So, with $a = 2$, $b = 4$, we get $(f^{-1})'(4) = \frac{1}{f'(2)}$. But $f'(x) = 3x^2$, so $f'(2) = 3 \cdot 2^2 = 3 \cdot 4 = 12$, i.e. $(f^{-1})'(4) = \boxed{\frac{1}{12}}$.

② $f(x) = x^5 + 2x + 1$.

In this case, solving for $f^{-1}(x)$ directly is too hard. So we have to use the inverse function theorem. We know we want $b = 4$, so try to solve $x^5 + 2x + 1 = 4$. Then

$x^5 + 2x - 3 = 0 \rightarrow (x-1)(x^4 + x^3 + x^2 + x + 3) = 0$ so $x = 1$ is a solution. So $a = 1$ and

$$(f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{5x^4 + 2}\Big|_{x=1} = \frac{1}{5 \cdot 1^4 + 2} = \frac{1}{5+2} = \boxed{\frac{1}{7}}$$