§6.3

Main Concepts

1. The number e.
2. Properties of exponentials

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1. You may have been concerned at the end of section §6.1: we showed that $\ln(x)$ has properties we expect ($\ln(1)=0$, $\ln(ab)=\ln(a)+\ln(b)$, 'domain $=(0,\infty)$', etc.), but these are properties that $\log_2(x)$ or $\log_{10}(x)$ also share. Why is $\ln(x)=\log_e(x)$ specifically? Answering this is a bit subtle: it depends on how you define "e." We'll take a simple route and say that e is the only number with $\boxed{\ln(e)=1.}$ I.e. $\int_1^e \frac{1}{t}dt = 1$.

Now, let's see if we can convince ourselves that $\ln(x)=\log_e(x)$ with this definition. First, we've seen that $f(x)=\ln(x)$ is invertible since it's monotonic since $f'(x)=\frac{1}{x}>0$ on the domain. So, there's some function $f^{-1}(x)$. Then

$$e^x = f^{-1}(f(e^x)) = f^{-1}(\ln(e^x)) = f^{-1}(x\cdot\ln(e)) = f^{-1}(x\cdot 1) = f^{-1}(x)$$

($f^{-1}(f(\text{anything}))=\text{anything}$; exponent rule for $\ln$; $\ln(e)=1$.)

I.e. $f^{-1}(x)=e^x$. So $\ln(x)$ has inverse e^x, i.e. it really is the logarithm with base e.

Summary: $\bullet\ \ln(e)=1$ $\quad\bullet\ \ln(e^x)=x$ $\quad\bullet\ e^{\ln(x)}=x$

2. Now we can recover properties of e^x using properties of $\ln(x)$. For example, if A and B are arbitrary numbers, then

$$e^{A+B} = e^{\ln(e^A)+\ln(e^B)} = e^{\ln(e^A\cdot e^B)} = e^A\cdot e^B$$

gives the usual rule for adding exponents. We can also prove calculus properties. For example, if $y=e^x$ then $x=\ln(y)$, and differentiating with respect to x gives $1=\frac{d}{dx}(\ln(y))=\frac{1}{y}\cdot\frac{dy}{dx}$ by the Chain rule. So $\frac{dy}{dx}=y=e^x$. In other words, e^x is its own derivative. Thus, it is also its own antiderivative:

Properties: $\bullet\ \frac{d}{dx}(e^x)=e^x$ $\quad\bullet\ \int e^x dx = e^x + C$.

Combined with other rules for derivatives and integrals, this opens up new problems.

Examples: Find $f'(x)$ for each of the following.

① $f(x)=e^x$.

We just did this: $f'(x)=e^x$.

② $f(x) = e^{12x}$

Use the chain rule: $f'(x) = e^{12x} \cdot (12) = 12e^{12x}$

③ $f(x) = e^{x^2+2x}$

Chain rule again: $f'(x) = e^{x^2+2x}(2x+2) = (2x+2)e^{x^2+2x}$

④ $f(x) = xe^x$

Product rule now: $f'(x) = (1)(e^x) + (x)(e^x) = (x+1)e^x$.

EXAMPLES: Evaluate each integral.

① $\int_0^3 e^x dx$

We know the antiderivative directly: $\int_0^3 e^x dx = e^x\Big|_0^3 = e^3 - e^0 = e^3 - 1$.

② $\int e^{4x} dx$

Use a substitution: $u = 4x$, $du = 4dx$, so

$$\int e^{4x} dx = \int e^u \cdot \frac{du}{4} = \frac{1}{4}\int e^u du = \frac{1}{4}e^u + C = \frac{1}{4}e^{4x} + C.$$

③ $\int_0^1 xe^{x^2+1} dx$

Substitute again: $u = x^2+1$, $du = 2x\,dx$, so the bounds convert $\begin{matrix} x=1 \to u=(1)^2+1=2 \\ x=0 \to u=(0)^2+1=1 \end{matrix}$ and

$$\int_0^1 xe^{x^2+1} dx = \int_1^2 xe^u \frac{du}{2x} = \frac{1}{2}\int_1^2 e^u du = \frac{1}{2}e^u\Big|_1^2 = \frac{1}{2}(e^2 - e^1) = \frac{e^2-e}{2}.$$