§6.4

Main Concepts

1. Exponential change-of-base
2. Logarithm change-of-base

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1. We now understand $\ln(x)$ and e^x, in the algebraic and calculus senses. But what about other bases? E.g. how do we differentiate or integrate 2^x or 10^x or $\log_{12}(x)$, etc.? For exponentials, the idea is to use two facts:

$$e^{\ln(x)} = x \qquad \text{and} \qquad (a^b)^c = a^{b\cdot c}$$

Through these, we can convert any base to base e.

Examples: Find $f'(x)$.

① $f(x) = 2^x$.

We rewrite $f(x) = 2^x = \left(e^{\ln(2)}\right)^x = e^{\ln(2)\cdot x}$. So, by the chain rule,

$$f'(x) = e^{\ln(2)\cdot x}\left(\ln(2)\right) = \boxed{2^x \cdot \ln(2)}$$

② $f(x) = 7^{x^2+1}$

Same idea: $f(x) = 7^{x^2+1} = \left(e^{\ln(7)}\right)^{x^2+1} = e^{\ln(7)(x^2+1)}$, so by the chain rule,

$$f'(x) = e^{\ln(7)(x^2+1)}\left(\ln(7)\cdot 2x\right) = \boxed{2x\ln(7)\cdot 7^{x^2+1}}$$

③ $f(x) = x^x$

We did this before with logarithmic differentiation, but our new technique also works:

$$f(x) = x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}, \text{ so } f'(x) = e^{x\ln(x)}\left((1)(\ln(x)) + (x)\left(\tfrac{1}{x}\right)\right)$$

$$\boxed{f'(x) = x^x\left(\ln(x)+1\right)}$$

Note: if you prefer to memorize, the first example works in general: $\boxed{\frac{d}{dx}a^x = a^x\ln(a)}$

Of course, if we can do derivatives, we can do some integrals, too.

Examples: Evaluate each integral.

① $\int 3^x dx$

Start by rewriting again: $\int 3^x dx = \int \left(e^{\ln(3)}\right)^x dx = \int e^{\ln(3)x} dx$. Then use the substitution $u = \ln(3)x$ so $du = \ln(3)dx$ and $\int e^{\ln(3)x} dx = \int e^u \frac{du}{\ln(3)} =$

$$= \frac{1}{\ln(3)}e^u + C = \frac{1}{\ln(3)}e^{\ln(3)x} + C = \boxed{\frac{1}{\ln(3)}3^x + C}$$

② $\int_0^1 x \cdot 5^{x^2}\,dx$

Start the same way: $5^{x^2} = \left(e^{\ln(5)}\right)^{x^2} = e^{\ln(5)x^2}$ so we'll use $u = \ln(5)x^2$ and $du = 2\ln(5)\,x\,dx$. Then $x=0 \to u=0$ and $x=1 \to u=\ln(5)$ and

$$\int_0^1 x\cdot 5^{x^2}dx = \int_0^1 x\,e^{\ln(5)x^2}dx = \int_0^{\ln(5)} e^u \frac{du}{2\ln(5)\cdot x} = \frac{1}{2\ln(5)}\int_0^{\ln(5)} e^u du$$

$$= \frac{1}{2\ln(5)} e^u\Big|_0^{\ln(5)} = \frac{1}{2\ln(5)}\left[e^{\ln(5)} - e^0\right] = \frac{1}{2\ln(5)}(5-1) = \boxed{\frac{2}{\ln(5)}}$$

2 We use the same approach with logarithms, coming from the change-of-base formula

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

EXAMPLE: Find $\frac{d}{dx}\log_7(x)$.

We rewrite using ln: $\frac{d}{dx}\log_7(x) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(7)}\right) = \frac{1}{\ln(7)}\frac{d}{dx}(\ln(x)) = \boxed{\frac{1}{\ln(7)}\cdot\frac{1}{x}}$

Again, the ideas of the last few examples work in general: $\boxed{\int a^x dx = \frac{a^x}{\ln(a)}}$ and

$$\boxed{\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}}$$