§6.5

Main Concepts

1. Exponential Growth / Decay
2. Newton's Law of Heating and Cooling

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1. Having seen first-order linear differential equations in general (see §6.6, we went out of order), we turn our attention to a specific interesting case. In many real-world situations, the rate at which a quantity increases/decreases is directly proportional to how much there currently is. E.g. in a population (of people, animals, bacteria, etc.) the more individuals there are, the more are born; in a bank account, the more money there is, the more interest it produces; and in a radioactive sample, the more radioactive material there is, the more loss of that material there is. If y is this quantity (population size, money, or mass of radioactive material), then our discussion above can be summarized by saying

$$\boxed{\frac{dy}{dt} = ky}$$

Literally: "the rate at which y changes is proportional to y." We can solve this using the techniques of §6.6: $y' = ky \rightarrow y' - ky = 0 \rightarrow y'e^{-kt} - kye^{-kt} = 0 \rightarrow (ye^{-kt})' = 0$

$$\rightarrow ye^{-kt} = \int 0\,dt = C \rightarrow \boxed{y = Ce^{kt}}$$

Because of the presence of e, we refer to such a system as experiencing <u>exponential growth</u> when $k > 0$ and <u>exponential decay</u> when $k < 0$.

Examples:

① A bacterial colony consists of 10000 organisms at noon and 40000 at 2pm. How many organism would you expect by 5pm at this growth rate?

Let's name our variables: y = # of organisms in 10000s,
t = hours after noon.

Then we have $y = Ce^{kt}$, $y(0) = 1$, and $y(2) = 4$. Plugging those in gives:

$$1 = Ce^{k(0)} = Ce^0 = C \cdot 1 = C \rightarrow C = 1 \quad \text{and}$$

$$4 = Ce^{k(2)} = (1)e^{2k} = e^{2k} \rightarrow 2k = \ln(4) \rightarrow k = \tfrac{1}{2}\ln(4) = \ln(2).$$

So, $y = Ce^{kt} = (1)e^{\ln(2)t} = (e^{\ln(2)})^t = 2^t$. So, $y(5) = 2^5 = 32$. I.e. we expect 320,000 organisms by 5pm.

② If a country's population went from 100 million to 140 million between 2020 and 2025, when will it reach 200 million?

Now, let y = population in millions.
t = years after 2020.

Then $y = Ce^{kt}$, $y(0) = 100$, and $y(5) = 140$.

Again, we plug these in:

$100 = Ce^{k(0)} = C \longrightarrow C = 100$

$140 = (100)e^{k(5)} \longrightarrow e^{5k} = 1.4 \longrightarrow 5k = \ln(1.4) \longrightarrow k = \frac{1}{5}\ln(1.4)$.

Then, we are asking: when is $y = 200$? I.e.

$$200 = 100e^{(\frac{1}{5}\ln(1.4))t} \longrightarrow e^{\frac{1}{5}\ln(1.4)t} = 2 \longrightarrow \frac{1}{5}\ln(1.4)t = \ln(2)$$

$$\longrightarrow t = \frac{5\ln(2)}{\ln(1.4)} \approx 10.3$$

So, we would expect to reach 200 million people some time during 2030–2031.

③ Carbon 14 is radioactive and decays. If you have a sample of ^{14}C, half of it will have decayed every 5730 years. If you start with 10 grams of ^{14}C, how much is left after 1000 years?

Let y = grams of ^{14}C.
t = years elapsed.

Then $y = Ce^{kt}$, $y(0) = 10$, and $y(5730) = 5$ (since half is gone at this point). So:

$10 = Ce^{k(0)} \longrightarrow C = 10$

$5 = 10e^{k(5730)} \longrightarrow 5730k = \ln(\frac{1}{2}) \longrightarrow k = -\frac{\ln(2)}{5730}$. So,

$y = 10e^{-\frac{\ln(2)}{5730}t}$ and $y(1000) = 10e^{-\frac{1000\ln(2)}{5730}} \approx 8.86$. I.e. 8.86 grams would remain.

④ Suppose a bank account accrues interest continuously at a rate of 4% yearly (APY). If it starts with \$300, how much money will it have after 2.5 years?

Let y = \$ in the account
t = time elapsed in years.

Then $y = Ce^{kt}$, $y(0) = 300$, and $y(1) = (1.04)\cdot 300$ (since it increased by 4%). So

$300 = Ce^{k(0)} = C \longrightarrow C = 300$

$(1.04)(300) = (300)e^{k(1)} \longrightarrow e^k = 1.04 \longrightarrow k = \ln(1.04)$.

So $y = 300e^{\ln(1.04)t} = 300(e^{\ln(1.04)})^t = 300(1.04)^t$. So, $y(2.5) = 300(1.04)^{2.5} \approx \$330.91$.

I.e. the account made approximately \$30.91 in that time.

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We'll end this section with another common application of ~~linear~~ first-order differential equations: Newton's Law of Heating and Cooling. The setup is as follows — an object is placed in an environment that is different from its own temperature. We expect that if we wait long enough, the object will reach the same temperature as the environment, but what happens in between? E.g. what is the temperature after 1 second or 1 minute or 12 minutes?

Newton's proposal is that if an object is much hotter than its environment, it will cool faster, and similarly if it is much colder. If T is the object's temperature, T_0 is the environment temperature,

and t is time, we can write this observation as:

$$\frac{dT}{dt} = k(T_0 - T) \qquad k>0 \text{ a constant.}$$

Indeed, if $T_0 > T$, $\frac{dT}{dt} = k(T_0 - T) > 0$, so the object heats up, and if $T_0 < T$, $\frac{dT}{dt} < 0$, so the object cools down. Let's solve this differential equation:

$$T' = kT_0 - kT \longrightarrow T' + kT = kT_0 \quad \text{so } P(t) = k \text{ and}$$

$I(t) = e^{\int k\,dt} = e^{kt}$ is our integrating factor. Then

$$T'e^{kt} + kTe^{kt} = kT_0e^{kt} \longrightarrow (Te^{kt})' = kT_0e^{kt}$$

$$\longrightarrow Te^{kt} = \int kT_0e^{kt}\,dt = T_0e^{kt} + C \longrightarrow \boxed{T = T_0 + Ce^{-kt}}$$

NEWTON'S LAW OF HEATING AND COOLING: An object with temperature T at time t sitting in an environment of constant temperature T_0 satisfies $T = T_0 + Ce^{-kt}$ for some constants $k>0$ and C.

EXAMPLE: A water bottle is sitting on a counter at 75°F when it is placed inside a refrigerator that maintains a temperature of 35°F. After 8 minutes, it has reached a temperature of 55°F. How cold is it after another 8 minutes? When does the water reach 40°F?

Let T = water bottle temperature (°F)

t = minutes after it was placed in the fridge

T_0 = fridge temperature = 35°F

Then we know that ⓘ $t=0$, $T=75$ and ⓘⓘ $t=8$, $T=55$. Plugging these into Newton's Law:

i) $75 = 35 + Ce^{-k(0)} \longrightarrow 40 = Ce^0 = C \longrightarrow C = 40.$

ii) $55 = 35 + Ce^{-k(8)} \longrightarrow 20 = Ce^{-8k} = (40)e^{-8k} \longrightarrow e^{-8k} = \frac{20}{40} = \frac{1}{2}$

$$\longrightarrow -8k = \ln(1/2) \longrightarrow k = -\tfrac{1}{8}\ln(1/2)$$

So $T = 35 + 40e^{\frac{1}{8}\ln(1/2)t}$. To answer the first question, plug in $t=16$ $(8+8)$:

$$T = 35 + 40e^{\frac{1}{8}\ln(1/2)\cdot 16} = 35 + 40e^{2\ln(1/2)} = 35 + 40\left(e^{\ln(1/2)}\right)^2 = 35 + 40\left(\tfrac{1}{2}\right)^2$$

$$= 35 + 40\left(\tfrac{1}{4}\right) = 35 + 10 = 45.$$

So after 8 more minutes, the water will be $\boxed{45°F}$

For the second question, plug in $T=40$:

$$40 = 35 + 40e^{\frac{1}{8}\ln(1/2)t} \longrightarrow 5 = 40\left(e^{\ln(1/2)}\right)^{1/8 t} \longrightarrow \tfrac{1}{8} = \tfrac{5}{40} = \left(\tfrac{1}{2}\right)^{\frac{1}{8}t} \longrightarrow$$

$$\longrightarrow \ln\left(\tfrac{1}{8}\right) = \tfrac{1}{8}t\ln\left(\tfrac{1}{2}\right) \longrightarrow t = \frac{8\ln\left(\frac{1}{8}\right)}{\ln\left(\frac{1}{2}\right)} = 24.$$

So the water will reach 40°F $\boxed{\text{24 minutes}}$ after it was placed in the fridge.