§6.6

Main Concepts

1. First-Order Linear Differential Equations
2. Integrating Factor
3. Particular Solutions

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1 Our next goal is to solve a class of differential equations. Recall that a differential equation is an equation involving the derivatives of an unknown function $f(x)$ or y. The class of differential equation we'll study is one that is first-order and linear. Here, "first-order" means the equation only includes the 1st derivative (for y' or $\frac{dy}{dx}$), but no higher derivatives ($f''(x)$, $f'''(x)$, y'''', $\frac{d^6y}{dx^6}$, etc.). "Linear" means linear in y, i.e. y only appears in derivatives and raised to the first power. So "y" and "$\frac{dy}{dx}$" are okay but y^2, $\cos(y)$, $\left(\frac{dy}{dx}\right)^2$, etc. are all not allowed.

In summary, a first-order linear differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{or} \quad y' + P(x)y = Q(x) \quad \text{or} \quad f'(x) + P(x)f(x) = Q(x).$$

Examples: Indicate whether each is a first-order, linear differential equation. If not, explain.

① $\frac{dy}{dx} = 0$

Yes! (This is $P(x) = Q(x) = 0$).

② $\frac{dy}{dx} = x^2 + 4$

Yes! ($P(x) = 0$, $Q(x) = x^2 + 4$).

③ $\frac{dy}{dx} = y$

Yes – it needs to be rearranged as $\frac{dy}{dx} - y = 0$ but then it is ($P(x) = -1$, $Q(x) = 0$)

④ $\frac{dy}{dx} = y^2$

No: can't have y^2 (not linear).

⑤ $y' = \cos(x)y + \sin(x)$

Yes: rearrange as $y' - \cos(x)y = \sin(x)$ and we're good ($P(x) = -\cos(x)$, $Q(x) = \sin(x)$)

⑥ $y'' = -y$.

No: no 2nd derivatives.

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2 Let's try to solve such a differential equation. The strategy is to see that the left-hand side of the equation almost looks like a product rule. Maybe if we multiply by some function $I(x)$, it will be exactly a product rule.

That is, we're hoping that $(y' + P(x)y)\,I(x) = (\text{product of something})'$.

One term of the product rule is $I(x)y'$ and the other is $I(x)P(x)y$. So, it makes sense that the "something" would be $I(x)\cdot y$. Then we have

$(I(x)y)' = I'(x)y + I(x)y' \overset{\text{want}}{=\!=} I(x)y' + I(x)P(x)y$. So, we want $I'(x) = I(x)P(x)$.

In other words, we want a function whose derivative gives itself times an extra factor of $P(x)$. But $I(x) = e^{\int P(x)\,dx}$ does exactly that: by the chain rule,

$$I'(x) = e^{\int P(x)\,dx} \cdot \left(\int P(x)\,dx\right)' = e^{\int P(x)\,dx} \cdot P(x) = I(x)P(x).$$

In summary, we have the following strategy:

SOLVING FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS:

ⓐ Write it, if needed, in the usual form: $y' + P(x)y = Q(x)$.

ⓑ Calculate the integrating factor $I(x) = e^{\int P(x)\,dx}$

ⓒ Multiply the differential equation by $I(x)$.

ⓓ Recognize the left-hand side as a product rule.

ⓔ Integrate both sides.

ⓕ Solve (algebraically) for y.

EXAMPLES: Solve each linear, first-order differential equation.

① $y' = -y$

Let's follow our steps: ⓐ Rewrite it as $y' + y = 0$ so $P(x) = 1$.

ⓑ Then $I(x) = e^{\int 1\,dx} = e^x$

ⓒ Multiply both sides by e^x: $y'e^x + ye^x = 0e^x$

$y'e^x + ye^x = 0$

ⓓ See the product rule: $(ye^x)' = 0$

ⓔ Integrate: $ye^x = \int 0\,dx = 0 + C = C$

ⓕ Solve for y: $ye^x = C$

$$\boxed{y = Ce^{-x}}$$

② $y' = 6 + 2y$

Ditto: ⓐ $y' - 2y = 6$ so $P(x) = -2$

ⓑ $I(x) = e^{\int -2\,dx} = e^{-2x}$

ⓒ $y'e^{-2x} - 2ye^{-2x} = 6e^{-2x}$

(d) $(ye^{-2x})' = 6e^{-2x}$

(e) $ye^{-2x} = \int 6e^{-2x}dx = -3e^{-2x} + C$

(f) $y = (-3e^{-2x} + C)\cdot e^{2x} = \boxed{-3 + Ce^{2x}}$

(3) $y' = \frac{3y}{x}$

Same idea: (a) $y' - \frac{3y}{x} = 0$ so $P(x) = -\frac{3}{x}$

(b) $I(x) = e^{-\int \frac{3}{x}dx} = e^{-3\ln(x)} = \left(e^{\ln(x)}\right)^{-3} = x^{-3}$

(c) $y'x^{-3} - 3yx^{-4} = 0$

(d) $(yx^{-3})' = 0$

(e) $yx^{-3} = \int 0\,dx = C$

(f) $\boxed{y = Cx^3}$

3 Notice that all of our solutions have an unknown constant. This is to be expected since we've taken one derivative and so lost one piece of data (the constant term). If we want to recover this, we need to provide an extra piece of data to compensate. This is usually given as an initial condition: one pair of (x,y) values that is known.

EXAMPLES: Solve each of the above examples with the initial condition $y(1) = 2$.

(1) We already got $y = Ce^{-x}$. Plugging in $x=1$, $y=2$ gives

$2 = Ce^{-1} \longrightarrow C = 2e$. So $y = 2e\cdot e^{-x} = \boxed{2e^{1-x}}$

(2) Similarly, $y = Ce^{2x} - 3$ so

$2 = Ce^{2\cdot 1} - 3 \longrightarrow 5 = Ce^2 \longrightarrow C = 5e^{-2}$ So $y = 5e^{-2}e^{2x} - 3$

$\boxed{y = 5e^{2x-2} - 3}$

(3) Here $y = Cx^3$ so

$2 = C\cdot 1^3 \longrightarrow C = 2$ so $\boxed{y = 2x^3}$

Without the initial condition, we found the general solution. After plugging in the initial condition and solving for y, we found the particular solution.