§6.7

Main Concepts

1. Slope fields
2. Euler's Method

---

1. Now that we understand linear first-order differential equations, let's see if we can understand things that are slightly more complicated. We'll remove the word "linear" but still restrict ourselves to differential equations of the form:

$$\frac{dy}{dx} = f(x,y)$$

In other words, our only restriction (other than first-order) is that we can solve for y' on one side of the equation.

Examples / Non-Examples:

These are all differential equations we can now consider:

(i) $y' = y^2$ (ii) $y' = x+y+\cos(y)$ (iii) $xy' + 2y = y^3$ (iv) $\ln(y'+y) = y^4+x$

[Note that examples (iii) and (iv) need some algebra first to solve for y']

These are NOT:

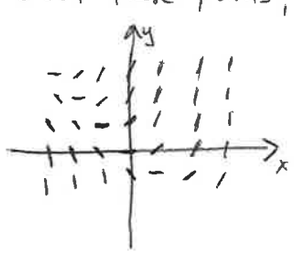
(i) $(y')^2 = x+y$ (ii) $\cos(y') = y^4-1$ (iii) $y'e^{y'} = 1$ (iv) $y'' = x-2y$

These differential equations can get quite complicated, and we won't be able to find explicit solutions to all of them. But we can still study them. Remember that the solution to the differential equation is a function: $y = (\text{something})$. Even if we can't solve it, we can ask: is it possible to graph it?

Looking at the diff. eq. ($y' = f(x,y)$) we see that it's telling us how to go from a point (x,y) on the graph to the slope y' at that point. Let's study $y' = x+y$ as an example. We could make a table:

x	-1	-1	-1	0	0	0	1	1	1	2	2	2	...
y	-1	0	1	-1	0	1	-1	0	1	-1	0	1	...
$y' = x+y$	-2	-1	0	-1	0	1	0	1	2	1	2	3	...

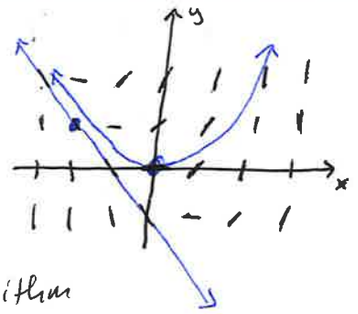
So, for example, the slope at $(-1,-1)$ is -2, and the slope at $(1,0)$ is 1. We can graph this: at each of these points, we'll draw a short line with the appropriate slope:



This graph is called a slope field.

Now, to actually draw a graph of a solution, we need an initial condition, eg. $y(0) = 0$ or $y(-2) = 1$. To draw the solution, start at the initial point and "follow" the slope field.

For the two examples given, we'd get (approximately) the two curves in blue.



2 This sketching process also gives an explicit algorithm for approximating solutions to differential equations like $y' = f(x,y)$: all we have to do is make the "follow" step more concrete. To do this, first pick a "step size", Δx. Then, if we start at a point (x,y), we know that a nearby point should be $(x+\Delta x, y+\Delta y)$, and that $\frac{\Delta y}{\Delta x} \approx$ slope $= f(x,y)$. So $\Delta y = f(x,y)\Delta x$.

This gives a process, called Euler's Method:

EULER'S METHOD: Given an initial value problem $y' = f(x,y)$ and $y(a) = b$, we can approximate points (x,y) on a solution as follows:

(i) Pick a step size Δx

(ii) Compute $y' = f(a,b)$.

(iii) Compute $\Delta y = f(a,b) \cdot \Delta x$

(iv) Repeat from step (ii) using the new point $(x+\Delta x, y+\Delta y)$.

EXAMPLES: Use Euler's method to approximate a solution to each initial value problem.

① $y' = y$ with $y(0) = 1$ using $\Delta x = 1$.

Note: we don't need to approximate this — it's first-order and linear, so we can solve it and would get $y = e^x$. This is good, because we can test our method's accuracy.

We'll follow the steps. For $(0,1)$, we get $y' = y = 1$ so $\Delta y = (1)\Delta x = (1)(1) = 1$.

So we get $(0,1) \longrightarrow (0+1, 1+1) = (1,2)$.

For $(1,2)$, we get $y' = y = 2$, so $\Delta y = 2$ and $(1,2) \longrightarrow (1+1, 2+2) = (2,4)$

For $(2,4)$, $y' = y = 4$, $\Delta y = 4$, so $\longrightarrow (3,8)$, etc.

In other words, we would approximate $\boxed{y(1) \approx 2,\ y(2) \approx 4,\ y(3) \approx 8}$ etc.

In reality, $y(1) = e^1 = e \approx 2.71828\ldots$, so our approximation is okay, but may be not great.

② $y' = y$ with $y(0) = 1$ using $\Delta x = 0.2$.

This is the same differential equation, but we've decreased the step size, and hopefully "increased the resolution" — getting a more accurate approximation. Let's calculate: $(x,y) = (0,1) \rightsquigarrow y' = y = 1 \rightsquigarrow \Delta y = (1)(0.2) = 0.2$

$\longrightarrow (x,y) = (0.2, 1.2) \rightsquigarrow y' = 1.2 \rightsquigarrow \Delta y = (1.2)(0.2) = 0.24$

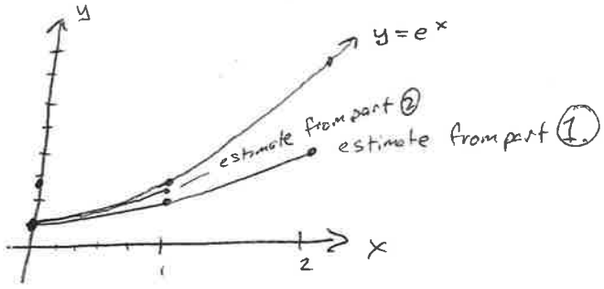
$\rightarrow (x,y) = (0.4,\ 1.44) \rightsquigarrow y' = 1.44 \rightsquigarrow \Delta y = (1.44)(0.2) = 0.288$

$\rightarrow (x,y) = (0.6,\ 1.728) \rightsquigarrow y' = 1.728 \rightsquigarrow \Delta y = 0.3456$

$\rightarrow (x,y) = (0.8,\ 2.0736) \rightsquigarrow y' = 2.0736 \rightsquigarrow \Delta y = 0.41472$

$\rightarrow (x,y) = (1,\ 2.48832)$

In other words, by shrinking Δx, we got a better estimate: $e \approx 2.48832$, which is much closer than $e \approx 2$. Graphically, we have:



③ $y' = 8y - y^2$ with $y(0) = 1$ using $\Delta x = 1$.

- NOTE: This isn't linear! So we actually can't (yet) solve this differential equation explicitly.

We'll compute:

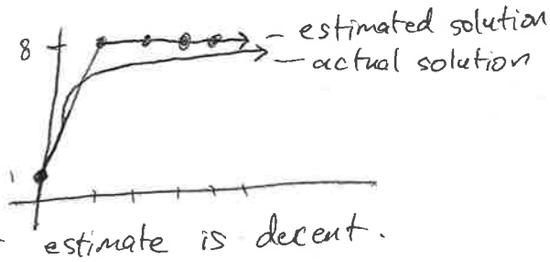
$(x,y) = (0,1) \rightsquigarrow y' = 8(1) - (1)^2 = 8 - 1 = 7 \rightsquigarrow \Delta y = (7)(1) = 7.$

$\rightarrow (x,y) = (1,8) \rightsquigarrow y' = 8(8) - (8)^2 = 64 - 64 = 0 \rightsquigarrow \Delta y = (0)(1) = 0.$

$\rightarrow (x,y) = (2,8) \rightsquigarrow y' = 8(8) - (8)^2 = 0 \rightsquigarrow \Delta y = 0$

$\rightarrow (x,y) = (3,8) \ldots$

We can see that the pattern will repeat: $y = 8$ so $y' = 0$ so $\Delta y = 0$, so the next point will also have $y = 8$. Graphically, we have the following, (with the actual solution having no explanation for now). Again, our estimate is decent.



One question we'd like to answer is how well Euler's method is doing. It seems to be true that a smaller Δx is better, but how small do we need? These kinds of questions are deep and require some extra thought and energy, but we can partially answer part of it: when does Euler's method overestimate vs. underestimate a solution?

Consider examples ① & ② above. Both underestimated the actual solution $y = e^x$. Why is this? The issue is that we only compute the slope every step. So we computed y' for $(x,y) = (0,1)$ and next computed it for $(x,y) = (0.2, 1.2)$, but what about in between? If we're between $(0,1)$ and $(0.2, 1.2)$, then $y' = y$ is between 1 and 1.2, so our slope ($y' = 1$) is an underestimate over this whole interval. This is the idea:

EULER'S METHOD ESTIMATION CRITERION: If Euler's method is applied to $y' = f(x,y)$

and one step goes from (a,b) to (c,d) such that $J(a,b) \leq J(x,y)$ for any (x,y) between (a,b) and (c,d), then (c,d) is an underestimate for the actual solution. Similarly, if $J(a,b) \geq J(x,y)$ for any (x,y) in between, then (c,d) is an overestimate. If neither is true ($J(x,y)$ is sometimes bigger, sometimes smaller) then we can't draw any conclusions.

EXAMPLES: Use the above criterion on the first step of Euler's method in each case.

① $y' = 1+y^2$, $y(0)=0$, $\Delta x = 0.1$.

The first step of Euler's method would give $y' = 1+(0)^2 = 1 \implies (0,0) \to (0.1, 0.1)$. If we plug in any (x,y) between $(0,0)$ and $(0.1, 0.1)$ into the formula for y', we'd get $y' = 1+(y)^2 > 1$ (since $y>0$), so our calculation of $y'=1$ is an underestimate.

② $y' = x+y^2$, $y(-1) = 1$, $\Delta x = 1$

For $(-1,1)$, we get $y' = (-1)+(1)^2 = -1+1 = 0$, so $\Delta y = (0)(1) = 0$ and our next point would be $(0,1)$. In between $(-1,1)$ and $(0,1)$, we'd get $y' = x+y^2 > (-1)+(1)^2 = 0$ so ours was again an underestimate.

③ $y' = x+y^2$, $y(-2) = 0$, $\Delta x = 1$

For $(x,y) = (-2,0) \implies y' = (-2)+(0)^2 = -1 \implies \Delta y = -1 \longrightarrow (0,0)$. But now, between $(-2,0)$ and $(0,0)$, things aren't so clear: @ $(x,y) = (0,0)$ $y' = 0$ is bigger
but @ $(x,y) = (-2,0)$ $y' = -2$ is smaller.

So it may be over- or under-estimating the solution. Note this is the same diff. eq., so the criterion also depends on the initial condition.