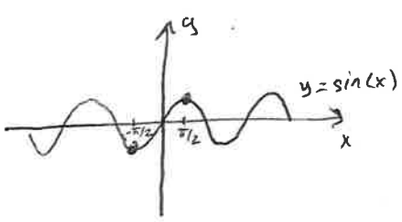
§6.8

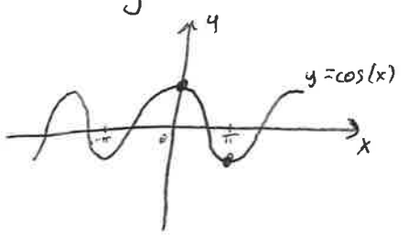
Main Concepts

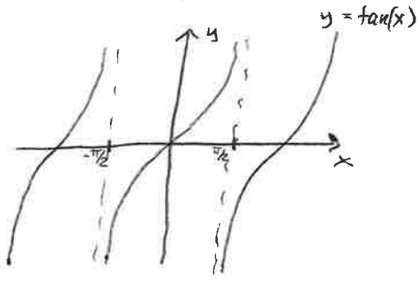
1. Inverse trig functions
2. Derivatives and integrals

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1. We've discussed derivatives and integrals for $\ln(x)$ and used this to understand its inverse function, e^x. We can do the same for trig functions. First, we recall the graphs:







Clearly, none of these pass the horizontal line test, so aren't invertible. Like before, we can resolve this by shrinking the domain.

Inverse Trig Functions:
- For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $\sin(x)$ is invertible with inverse $\arcsin(x)$,
- For $0 \le x \le \pi$, $\cos(x)$ is invertible with inverse $\arccos(x)$, and
- For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\tan(x)$ is invertible with inverse $\arctan(x)$.

(We can do something similar for arcsec, arccsc, and arccot, but we won't use them at all).

In other words, $\arcsin(x)$ is a number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

Examples: Compute the following quantities.

① $\arcsin(0)$

We know $\sin(-2\pi) = 0$, $\sin(-\pi) = 0$, $\sin(0) = 0$, $\sin(\pi) = 0$, $\sin(2\pi) = 0$, ... The only one in the correct domain is 0, so $\arcsin(0) = 0$.

② $\arccos\left(\frac{1}{2}\right)$

Again, we know $\cos\left(\pm\frac{\pi}{3}\right) = \frac{1}{2}$, $\cos\left(\pm\frac{7\pi}{3}\right) = \frac{1}{2}$, $\cos\left(\pm\frac{13\pi}{3}\right) = \frac{1}{2}$, ... The only one between 0 and π is $\pi/3$, so $\arccos\left(\frac{1}{2}\right) = \pi/3$.

③ $\arctan(1)$

Same idea: $\tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{9\pi}{4}\right) = \ldots = 1$ but the only one between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is $\pi/4$, so $\arctan(1) = \pi/4$.

④ $\sin(\arcsin(0.8))$

Let's write $x = \arcsin(0.8)$. Then $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $\sin(x) = 0.8$. But we are trying to find $\sin(\arcsin(0.8)) = \sin(x) = 0.8$ as we just said. This makes sense; the functions are inverses.

⑤ $\arcsin(\sin(3.4))$

We might be tempted to say these are inverses so $\arcsin(\sin(3.4)) = 3.4$, but this is

Incorrect: arcsin is only the inverse of $\sin(x)$ for $-\pi/2 \le x \le \frac{\pi}{2}$. So we need to use periodicity. So,

$$\arcsin(\sin(3.4)) = \arcsin(\sin(\pi - 3.4)) = \pi - 3.4$$

since $-\frac{\pi}{2} \le \pi - 3.4 \le \frac{\pi}{2}$.

⑥ $\arctan(\tan(21))$

Same idea: $\arctan(\tan(21)) = \arctan(\tan(21 - 7\pi)) = 21 - 7\pi$ since $-\frac{\pi}{2} < 21 - 7\pi < \frac{\pi}{2}$.

In summary of these last two examples:

- $\sin(\arcsin(x)) = x$ for $-1 \le x \le 1$
- $\cos(\arccos(x)) = x$ for $-1 \le x \le 1$
- $\tan(\arctan(x)) = x$ for all x.
- $\arcsin(\sin(x)) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
- $\arccos(\cos(x)) = x$ for $0 \le x \le \pi$
- $\arctan(\tan(x)) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(The conditions on the left are only the restrictions on the domain: essentially they say $\sin(\arcsin(x)) = x$ for any x that makes sense, while the conditions on the right are actually meaningful.)

NOTATION: Other texts use $\sin^{-1}(x)$ for $\arcsin(x)$. This is probably a better choice for analogy with $f^{-1}(x)$. However, it doesn't align with notation like $\sin^2(x)$, since $\sin^2(x) = (\sin(x))^2$, but $\sin^{-1}(x) \neq (\sin(x))^{-1}$! So, I will avoid the "$\sin^{-1}$" notation.

2 Now that we have the algebra in hand, we proceed to the calculus. Differentiating $\sin(\arcsin(x)) = x$ using the chain rule gives: $\cos(\arcsin(x)) \cdot \arcsin'(x) = 1$, so $\arcsin'(x) = \frac{1}{\cos(\arcsin(x))}$. This is accurate, but not very meaningful, so we use a trig identity to simplify: $\cos(\arcsin(x)) = \sqrt{\cos^2(\arcsin(x))} = \sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2}$.

So, we conclude $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$. We can do the same for the other inverse trig functions. The most important two are:

INVERSE TRIG DERIVATIVES: • $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ $(-1 \le x \le 1)$ • $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

As always, we can combine these with previous techniques to solve more problems.

EXAMPLES: Find the specified derivatives.

① $\frac{d}{dx}(x\arcsin(x^2))$

This has a product and chain rule:

$$\frac{d}{dx}(x\arcsin(x^2)) = 1 \cdot \arcsin(x^2) + x \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot (2x) = \arcsin(x^2) + \frac{2x^2}{\sqrt{1-x^4}}$$

② $f'(0)$ for $f(x) = e^{\arctan(x)}$

By chain rule: $f'(x) = e^{\arctan(x)} \cdot \frac{1}{1+x^2}$, so $f'(0) = e^{\arctan(0)} \cdot \frac{1}{1+0^2} = e^0 \cdot 1 = 1$.

③ $(f^{-1})'(1)$ for $f(x) = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

One approach is to recognize that because of the domain restriction, $f^{-1}(x) = \arctan(x)$, so $(f^{-1})'(x) = \frac{1}{1+x^2}$ and $(f^{-1})'(1) = \frac{1}{1+1^2} = \frac{1}{2}$.

Alternatively, we can get here using the inverse function theorem. First, solve $f(x) = 1 \longrightarrow$ $\rightarrow \tan(x) = 1 \longrightarrow x = \frac{\pi}{4}$. Then $(f^{-1})'(1) = \frac{1}{f'(\pi/4)} = \frac{1}{\sec^2(\pi/4)} = \cos^2(\pi/4) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

Of course, we can now solve some integrals, too:

EXAMPLES: Evaluate each integral.

① $\int \frac{1}{\sqrt{9-x^2}}\,dx$

This almost looks like "arcsin'," so let's factor out the 9 to get closer:

$$\frac{1}{\sqrt{9-x^2}} = \frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}} = \frac{1}{3\sqrt{1-\frac{x^2}{9}}} = \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}.$$

So, this is close to arcsin if $\frac{x}{3}$ is a single variable: let $u = \frac{x}{3}$, $du = \frac{dx}{3}$, so

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{3\sqrt{1-(x/3)^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}} \cdot 3\,du = \arcsin(u) + C = \arcsin\left(\frac{x}{3}\right) + C$$

② $\int \frac{1}{x^2+2x+2}\,dx$

Same idea as the previous one, but without the "$\sqrt{\ }$" this is more like arctan. To get this form, we need more algebra than in the previous one: here we complete the square.

$$x^2+2x+2 = (x^2+2x+1)+1 = 1+(x+1)^2 \qquad \text{so if } u = x+1,\ du = dx, \text{ and}$$

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{1+(x+1)^2}\,dx = \int \frac{1}{1+u^2}\,du = \arctan(u) + C = \arctan(x+1) + C.$$