§6.9

MAIN CONCEPT

II Hyperbolic trig functions

I For trigonometry, we've developed functions (sin, cos, tan, sec, ...) to help us convert between algebra, calculus, and geometry. There is a similar set of functions (sinh, cosh, tanh, sech, ...) for "different geometries," called hyperbolic trigonometric functions. This course just introduces them, so for now, all I need you to know is the following facts/definitions:

HYPERBOLIC TRIG FUNCTIONS: The hyperbolic trig functions are defined as:

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$
- $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$
- $\operatorname{csch}(x) = \frac{1}{\sinh(x)}$
- $\operatorname{sech}(x) = \frac{1}{\cosh(x)}$
- $\coth(x) = \frac{1}{\tanh(x)}$.

Just as with trig, we have the "Pythagorean theorem":

HYPERBOLIC PYTHAGOREAN IDENTITY: For all x, $\cosh^2(x) - \sinh^2(x) = 1$.

Finally, using the definitions/identity above, we can work out the derivatives. The most important are:

HYPERBOLIC TRIG DERIVATIVES: • $\frac{d}{dx}\sinh(x) = \cosh(x)$ • $\frac{d}{dx}\cosh(x) = \sinh(x)$ • $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$

Of course, we can use all of these in problems:

EXAMPLES: Compute the indicated values.

① $\cosh(0)$

By definition, $\cosh(0) = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$.

② $f'(1)$ for $f(x) = 2\sinh(x^2)$.

By the chain rule, $f'(x) = 2\cosh(x^2)\cdot(2x)$ so $f'(1) = 2\cosh(1^2)\cdot(2(1)) = 4\cdot\frac{e^1 + e^{-1}}{2}$

$= 2(e + 1/e)$.

③ $\int_0^1 \tanh(x)\,dx$

Rewrite using the definition: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$. This suggests using $u = \cosh(x)$, $du = \sinh(x)dx$, with the new bounds $x=0 \to u = \cosh(0) = 1$ and $x = 1 \to u = \cosh(1) = \frac{e+e^{-1}}{2}$. So

$$\int_0^1 \tanh(x)\,dx = \int_1^{\frac{e+e^{-1}}{2}} \frac{\sinh(x)}{u}\cdot\frac{du}{\sinh(x)} = \int_1^{\frac{e+e^{-1}}{2}} \frac{du}{u} = \ln|u|\Big|_1^{\frac{e+e^{-1}}{2}} = \ln\left(\frac{e+e^{-1}}{2}\right).$$