§7.2

MAIN CONCEPT

II Integration by parts

I For differentiation, we have a product rule: $d(uv) = du \cdot v + u \cdot dv$. We don't have a product rule for integrals, but integrating the above gives:

$$uv = \int v\,du + \int u\,dv \longrightarrow \boxed{\int u\,dv = uv - \int v\,du}$$

This formula is some kind of "antiproduct rule", called integration by parts. Let's use it to solve problems; the idea is to write an given integral in the form $\int u\,dv$, where "dv" is something we can integrate, and so that "$\int v\,du$" is an easier integral.

EXAMPLES: Use integration by parts.

① $\int x\cos(x)\,dx$

This isn't an integral we know, and no substitution looks helpful. So, let's integrate by parts. If we choose $u = x$, $dv = \cos(x)\,dx$ then we can find du and v: $u = x \to du = dx$; $v = \sin(x) \leftarrow dv = \cos(x)\,dx$.

(notice the organization; this may help you). Then, we have

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = \boxed{x\sin(x) + \cos(x) + C}$$

The advantage of integration by parts here is that $du = 1\,dx$ is simpler than $u = x$.

② $\int x^2 \ln(x)\,dx$

Here, we see that $\ln(x)$ gets much simpler by differentiating, so we choose $u = \ln(x) \to du = \frac{1}{x}dx$; $v = \frac{1}{3}x^3 \leftarrow dv = x^2\,dx$.

So: $\int x^2\ln(x)\,dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx = \frac{1}{3}x^3\ln(x) - \frac{1}{3}\int x^2\,dx$

$$= \boxed{\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C}$$

③ $\int \arctan(x)\,dx$

At first, this looks odd: there are no "parts" for integration by parts. But $\arctan(x)$ does become simpler on differentiating, so we'll choose $u = \arctan(x) \to du = \frac{1}{1+x^2}dx$; $v = x \leftarrow dv = dx$. Then,

$\int \arctan(x)\,dx = x\cdot\arctan(x) - \int \frac{x}{1+x^2}\,dx$. For this second integral, we can use the substitution $u = 1+x^2 \to du = 2x\,dx$ [NOTE: I probably shouldn't reuse the letter "u" here; if you prefer, write $w = 1+x^2 \to dw = 2x\,dx$]. Then,

$$\int \frac{x}{1+x^2}dx = \int \frac{x}{u}\cdot\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+x^2| + C.$$

Putting it all together, we have: $\int \arctan(x)dx = \boxed{x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$

(4) $\int x^2\cos(x)dx$

Here "x^2" is simplified by differentiating and "cos(x)" isn't too bad to integrate so $u = x^2 \to du = 2xdx$, $v = \sin(x) \leftarrow dv = \cos(x)dx$

$\int x^2\cos(x)dx = x^2\sin(x) - \int(\sin(x))(2x)dx$ The new integral isn't immediately obvious, but it looks like example (1), so let's integrate by parts again: $u = 2x \to du = 2dx$, $v = -\cos(x) \leftarrow dv = \sin(x)dx$

$$= x^2\sin(x) - \left[(2x)(-\cos(x)) - \int(-\cos(x))(2dx)\right]$$

$$= x^2\sin(x) + 2x\cos(x) - 2\int\cos(x)dx$$

$$= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

In other words, it can be useful to integrate by parts repeatedly.

(5) $\int e^x\sin(x)dx$.

Let's try $u = \sin(x) \to du = \cos(x)dx$, $v = e^x \leftarrow dv = e^xdx$ to get $\int e^x\sin(x)dx = e^x\sin(x) - \int e^x\cos(x)dx$.

We have a similar integrand so try again: $u = \cos(x) \to du = -\sin(x)dx$, $v = e^x \leftarrow dv = e^xdx$ to get

$$\int e^x\sin(x)dx = e^x\sin(x) - \left[e^x\cos(x) - \int e^x(-\sin(x))dx\right]$$

$$\int e^x\sin(x)dx = e^x\sin(x) - e^x\cos(x) - \int e^x\sin(x)dx$$

At first, this seems unhelpful — we're back where we started! But this is good because now we can solve the problem algebraically. Even though "$\int e^x\sin(x)dx$" is complicated, it's still just a function, so adding it to both sides gives

$$\int e^x\sin(x)dx + \int e^x\sin(x)dx = e^x\sin(x) - e^x\cos(x)$$

$$2\int e^x\sin(x)dx = e^x\sin(x) - e^x\cos(x)$$

$$\boxed{\int e^x\sin(x)dx = \frac{e^x\sin(x) - e^x\cos(x)}{2} + C}$$