§7.3

MAIN CONCEPTS

1. Trig identities
2. Application to some integrals.

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1. The most fundamental trigonometric identity is the Pythagorean identity

$$\cos^2(x) + \sin^2(x) = 1.$$

Dividing through by $\cos^2(x)$ and $\sin^2(x)$, respectively, gives two variations:

$$1 + \tan^2(x) = \sec^2(x) \qquad \text{and} \qquad \cot^2(x) + 1 = \csc^2(x).$$

These are useful to convert $\sin^2 \leftrightarrow \cos^2$ or $\tan^2 \leftrightarrow \sec^2$ or $\cot^2 \leftrightarrow \csc^2$ when computing integrals.

Another useful identity is with a sum of angles:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \quad ; \quad \sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b).$$

We can use these in two useful ways. First, by taking $a = b = x$ in the cosine formula, we get $\cos(2x) = \cos^2(x) - \sin^2(x)$. By using Pythagores, we get

$$\begin{cases} \cos(2x) = (1-\sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x) \\ \text{AND} \quad \cos(2x) = \cos^2(x) - (1-\cos^2(x)) = 2\cos^2(x) - 1. \end{cases}$$

These two identities can be useful in computing integrals because you can decrease the exponent $\sin^2(x) \longleftrightarrow \cos(2x)$ $\qquad \cos^2(x) \longleftrightarrow \cos(2x)$.

Finally, the sum of angles can be used another way. Plugging in "$-b$" for b gives

$$\cos(a-b) = \cos(a)\cos(-b) - \sin(a)\sin(-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\sin(a-b) = \cos(a)\sin(-b) + \sin(a)\cos(-b) = -\cos(a)\sin(b) + \sin(a)\cos(b).$$

Adding these to the original formulas gives

$$\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$$

$$\cos(a+b) - \cos(a-b) = -2\sin(a)\sin(b)$$

$$\sin(a+b) + \sin(a-b) = 2\sin(a)\cos(b).$$

These are useful because you can change a product [cos·cos, sin·sin, sin·cos] into a sum [cos+cos, cos−cos, sin+sin], which is easier to integrate.

2. Let's apply these identities to some examples.

EXAMPLES: Compute each integral

① $\int \sin^4(x)\cos^3(x)\,dx$

This looks like a u-sub, but both $u=\sin(x)$ and $u=\cos(x)$ leaves "extra" terms. But if we split up $\cos^3(x) = \cos^2(x)\cdot\cos(x)$, then Pythagoras ~~gives~~ lets us rewrite this as $= (1-\sin^2(x))\cos(x)$ so there's no "extra" cosine. Take $u=\sin(x)$ $du=\cos(x)dx$

$$\int \sin^4(x)\cos^3(x)dx = \int \sin^4(x)(1-\sin^2(x))\cos(x)\,dx = -\int u^4(1-u^2)\,du$$

$$= \int (u^4-u^6)\,du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + C$$

② $\int \sin^5(x)\cos^2(x)dx$

Same idea: if we pull out $\sin^4(x) = (\sin^2(x))^2 = (1-\cos^2(x))^2$ then we're set up for a u-sub since we have just one $\sin(x)$ left. Let $u=\cos(x)$ $du=-\sin(x)dx$ to get:

$$\int \sin^5(x)\cos^2(x)dx = \int \sin^4(x)\cdot\cos^2(x)\cdot\sin(x)dx = \int (1-\cos^2(x))^2\cdot\cos^2(x)\cdot\sin(x)dx$$

$$= \int (1-u^2)^2\cdot u^2\cdot(-du) = -\int (1-2u^2+u^4)\cdot u^2\,du$$

$$= -\int (u^2-2u^4+u^6)\,du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= -\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$$

③ $\int \sin^2(x)dx$

Now, the idea stops working. We'd like to "pull out" $\sin(x)$ to leave $\sin(x)dx$ for the u-sub, but the lone $\sin(x)$ can't be rewritten by Pyth. since it's an odd exponent (we had $\cos^2(x)$ and $\sin^4(x)$ before). So, instead we use the double-angle identity $\cos(2x) = 1-2\sin^2(x)$ to reduce the exponent. Solving for $\sin^2(x)$ gives

$\sin^2(x) = \frac{1-\cos(2x)}{2}$, so:

$$\int \sin^2(x)dx = \int \frac{1-\cos(2x)}{2}dx = \int \frac{1}{2}dx - \frac{1}{2}\int \cos(2x)dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

④ $\int \sin^2(x)\cos^2(x)dx$

Same idea: Pythagoras doesn't help much because both exponents are even, so we use both double-angle identities. The other one is $\cos(2x) = 2\cos^2(x) - 1$ or $\cos^2(x) = \frac{1+\cos(2x)}{2}$.

So: $\int \sin^2(x)\cos^2(x)dx = \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)dx = \frac{1}{4}\int (1-\cos^2(2x))\,dx.$

We're left with another even exponent, so do it again:

$$= \frac{1}{4}\int \left(1 - \frac{1+\cos(4x)}{2}\right)dx = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$

⑤ $\int \sin(x)\sin(4x)\,dx$

As suggested in section [1], we have a strategy here to convert a product (sin·sin) into a sum (cos − cos). Using that identity with $a=x$ and $b=4x$ gives:

$$-2\sin(x)\sin(4x) = \cos(x+4x) - \cos(x-4x) = \cos(5x) - \cos(-3x) = \cos(5x) - \cos(3x).$$

So, dividing by -2 and integrating gives:

$$\int \sin(x)\sin(4x)\,dx = -\frac{1}{2}\int (\cos(5x) - \cos(3x))\,dx = -\frac{1}{10}\sin(5x) + \frac{1}{6}\sin(3x) + C.$$

⑥ $\int \tan^3(x)\,dx$

When we're unsure, converting back to sine and cosine can be helpful. Here, this gives

$$\int \frac{\sin^3(x)}{\cos^3(x)}\,dx = \int \frac{(1-\cos^2(x))}{\cos^3(x)} \cdot \sin(x)\,dx$$ which is setup for the substitution $u = \cos(x)$.

For practice, finish this integral, but for now I'll demonstrate another approach using the modified Pythagorean identity:

$$\int \tan^3(x)\,dx = \int \tan(x)\cdot\tan^2(x)\,dx = \int \tan(x)\left(\sec^2(x) - 1\right)dx = \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx$$

This is useful because with $u = \tan(x)$, $du = \sec^2(x)\,dx$ and this becomes

$$= \int u\cdot du - \int \tan(x)\,dx = \frac{1}{2}u^2 + \ln|\cos(x)| + C$$

$$= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C$$

(Note: we have $\int \tan(x)\,dx = -\ln|\cos(x)| + C$ memorized as per §7.1)

⑦ $\int \frac{1}{\cos^2(x)}\,dx$

Sometimes, when we're stuck, it can help to just rewrite things. Here, we have

$$\int \frac{1}{\cos^2(x)}\,dx = \int \sec^2(x)\,dx = \tan(x) + C$$

just from memory.

There were lots of examples here, but roughly, the summary is:

- $\int \sin^n(x)\cos^m(x)\,dx$ → if either n or m is odd, use Pythagoras and u-substitution.
  → if both n and m are even, use double-angle identity.
- $\int \sin(ax)\cos(bx)$ or other products with different angles ("ax" vs "bx") → Angle sum identities.
- Other trig integrals → Convert to sines and cosines OR
  → Try other trig identities OR
  → Recognize the integral.