§7.4

MAIN CONCEPTS

1. Roots of linear functions
2. Roots of quadratics
3. Completing the square.

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1. As a general hint, if an integral involves $\sqrt[n]{ax+b}$ (i.e. the n^{th} root of something linear) then it often helps to take $u = \sqrt[n]{ax+b}$ for a substitution. Our next step would usually be computing du, but some algebra helps here: $u^n = ax+b \longrightarrow nu^{n-1}du = a\,dx$. This is nice because it instantly converts dx into all "u"s. Let's use this:

EXAMPLES! Compute each integral.

(1) $\int \frac{dx}{x+\sqrt{x}}$

As suggested, use $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u} = 2\int \frac{u\,du}{u(u+1)} = 2\int \frac{du}{u+1} = 2\ln|u+1| + C$$
$$= 2\ln|\sqrt{x}+1| + C.$$

(2) $\int_{-1}^{0} x\sqrt[3]{2x+1}\,dx$

Now use $u = \sqrt[3]{2x+1}$ so $u^3 = 2x+1$ and $3u^2du = 2dx$. We also need $x = \frac{u^3-1}{2}$.

So to plug in, and our bounds change: $x=0 \longrightarrow u = \sqrt[3]{2\cdot 0+1} = 1$, $x=-1 \longrightarrow u = \sqrt[3]{2\cdot(-1)+1} = -1$. So:

$$\int_{-1}^{0} x\sqrt[3]{2x+1}\,dx = \int_{-1}^{1} \left(\frac{u^3-1}{2}\right)\cdot u \cdot \left(\frac{3}{2}u^2du\right) = \frac{3}{4}\int_{-1}^{1}(u^3-1)\cdot u^{\bullet}\cdot u^2\,du$$
$$= \frac{3}{4}\int_{-1}^{1}(u^6-u^3)\,du = \frac{3}{28}u^7\Big|_{-1}^{1} - \frac{3}{16}u^4\Big|_{-1}^{1} = \frac{3}{28}\left(1^7-(-1)^7\right) - \frac{3}{16}\left(1^4-(-1)^4\right) = \frac{3}{14}.$$

(3) $\int \frac{x}{(3x-4)^{2/3}}\,dx$

Note that $(3x-4)^{2/3} = \left[(3x-4)^{1/3}\right]^2 = \left[\sqrt[3]{3x-4}\right]^2$. So the suggestion here is to take $u = \sqrt[3]{3x-4}$ (without the "squared") so $u^3 = 3x-4$ and $3u^2du = 3\,dx$ i.e. $dx = u^2du$. Then $x = \frac{u^3+4}{3}$ so we get:

$$\int \frac{x}{(3x-4)^{2/3}}\,dx = \int \frac{\left(\frac{u^3+4}{3}\right)}{u^2}\cdot u^2du = \frac{1}{3}\int (u^3+4)\,du = \frac{1}{12}u^4 + \frac{4}{3}u + C$$
$$= \frac{1}{12}(3x-4)^{4/3} + \frac{4}{3}(3x-4)^{1/3} + C$$

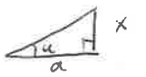
2. Another common case is the square root of a quadratic. We've seen one of these already: $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$. This hints at a more general strategy called trig substitution. The idea is that if $x = \sin(u)$, then $\sqrt{1-x^2} = \sqrt{1-\sin^2(u)}$

$= \sqrt{\cos^2(u)} = |\cos(u)|$ so we were able to eliminate the square root. More generally:

TRIG SUBSTITUTION: If you see an integral including ↙ ; try the substitution ↙



$\sqrt{a^2-x^2} \longrightarrow x = a\cdot\sin(u)$



$\sqrt{a^2+x^2} \longrightarrow x = a\cdot\tan(u)$

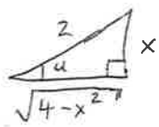


$\sqrt{x^2-a^2} \longrightarrow x = a\cdot\sec(u)$

Furthermore, at the end of such a problem, to solve expressions in terms of x, draw a right triangle corresponding to the substitution.

EXAMPLES: Compute each integral.

① $\int \sqrt{4-x^2}\,dx$

As suggested, substitute $x = 2\sin(u)$, so $dx = 2\cos(u)du$ and 

. Then

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-(2\sin(u))^2}\cdot(2\cos(u))\,du = \int \sqrt{4-4\sin^2(u)}\cdot 2\cos(u)\,du = 4\int \sqrt{1-\sin^2(u)}\cos(u)\,du$$

$$= 4\int \sqrt{\cos^2(u)}\cos(u)\,du = 4\int \cos(u)\cdot\cos(u)\,du = 4\int \cos^2(u)\,du.$$

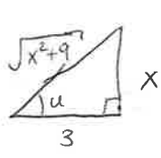
We can solve this with the techniques of last section:

$$= 4\int \left(\frac{1+\cos(2u)}{2}\right)du = 2u + \sin(2u) + C = 2u + 2\sin(u)\cos(u) + C.$$

Then we can solve in terms of x using the triangle: $u = \arcsin\left(\frac{x}{2}\right)$, $\sin(u) = \frac{x}{2}$, and $\cos(u) = \frac{\sqrt{4-x^2}}{2}$

So: $\int \sqrt{4-x^2}\,dx = 2\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C$

② $\int \frac{dx}{(x^2+9)^{3/2}}$

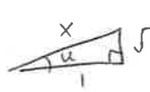
Now the suggestion is $x = 3\tan(u)$, so $dx = 3\sec^2(u)\,du$ and 

. So,

$$\int \frac{dx}{(x^2+9)^{3/2}} = \int \frac{3\sec^2(u)}{(9\tan^2(u)+9)^{3/2}}\,du = \frac{3}{9^{3/2}}\int \frac{\sec^2(u)}{(\tan^2(u)+1)^{3/2}}\,du = \frac{3}{27}\int \frac{\sec^2(u)}{\sec^3(u)}\,du$$

$$= \frac{1}{9}\int \cos(u)\,du = \frac{1}{9}\sin(u) + C = \frac{1}{9}\cdot\frac{x}{\sqrt{x^2+9}} + C.$$

③ $\int \frac{2dx}{x^2-1}$

Use $x = \sec(u)$, $dx = \sec(u)\tan(u)\,du$, and 

to get

$$\int \frac{2dx}{x^2-1} = 2\int \frac{\sec(u)\tan(u)}{\sec^2(u)-1}\,du = 2\int \frac{\sec(u)\tan(u)}{\tan^2(u)}\,du = 2\int \frac{\sec(u)}{\tan(u)}\,du = 2\int \frac{\frac{1}{\cos(u)}}{\frac{\sin(u)}{\cos(u)}}\,du$$

$$= 2\int \frac{1}{\sin(u)}\,du = 2\int \csc(u)\,du = 2\ln\left|\csc(u) - \cot(u)\right| + C$$

$= 2\ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| + C = 2\ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right|^2 + C$

$= \ln\left|\frac{(x-1)^2}{x^2-1}\right| + C = \ln\left|\frac{(x-1)(x+1)}{(x-1)(x+1)}\right| + C = \ln\left|\frac{x-1}{x+1}\right| + C.$

(We'll see another approach to this in §7.5)

3 Finally, we can see that this doesn't cover all quadratics; what about something like $\sqrt{x^2+2x+2}$? In this case, we can convert it into the usual cases by <u>completing the square</u>. Recall that we do this by writing

$$ax^2+bx+c = a\left(x-\frac{b}{2a}\right)^2 + \cdots$$

and filling in the "...". The idea is that expanding $a(x-\frac{b}{2a})^2$ gives $ax^2+bx+\cdots$ and we just need to fix the constant.

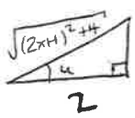
- <u>EXAMPLES</u>: Integrate by completing the square and using a trig substitution.

① $\int \frac{dx}{4x^2+4x+5}$

We start by completing the square: $4x^2+4x+5 = 4(x^2+x)+5 = 4\left(x^2+x+\frac{1}{4}\right)+5-4\cdot\frac{1}{4}$

$= 4\left(x+\frac{1}{2}\right)^2 + 4 = (2x+1)^2+4$

So we have $\int \frac{dx}{(2x+1)^2+4}$. This suggests the trig. sub. $2x+1 = 2\tan(u)$, so $2dx = 2\sec^2(u)du$.

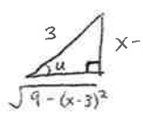


Then $\int \frac{dx}{(2x+1)^2+4} = \int \frac{\sec^2(u)du}{(2\tan(u))^2+4} = \int \frac{\sec^2(u)}{4(\tan^2(u)+1)}du = \frac{1}{4}\int \frac{\sec^2(u)}{\sec^2(u)}du = \frac{1}{4}\int du$

$= \frac{1}{4}u + C = \boxed{\frac{1}{4}\arctan\left(\frac{2x+1}{2}\right) + C}$

② $\int \frac{dx}{\sqrt{6x-x^2}}$

Again, start by completing the square: $6x-x^2 = -(x^2-6x) = -(x^2-6x+9)+9 = 9-(x-3)^2$.

So; we'll use $x-3 = 3\sin(u)$ giving $dx = 3\cos(u)du$ and 

. So:

$\int \frac{dx}{\sqrt{6x-x^2}} = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \int \frac{3\cos(u)du}{\sqrt{9-9\sin^2(u)}} = \frac{3}{\sqrt{9}}\int \frac{\cos(u)}{\sqrt{1-\sin^2(u)}}du = \frac{3}{3}\int \frac{\cos(u)}{\sqrt{\cos^2(u)}}du =$

$= \int \frac{\cos(u)}{\cos(u)}du = \int du = u + C = \boxed{\arcsin\left(\frac{x-3}{3}\right) + C}$