§7.5

MAIN CONCEPTS

1. Partial fraction decomposition
2. Integrating rational functions
3. Improper fractions

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1. Recall that a rational function is a ratio of polynomials. So each of these is rational:

$$\frac{x+1}{x+2}, \quad \frac{x^2-2x}{x^3+7}, \quad 4x-7, \quad x^3+x+1, \quad \frac{1}{x^2+2x-5}, \quad \frac{x^3+5x-1}{x^2+2}$$

but none of these are:

$$e^x, \quad \ln\left(\frac{x+1}{x-1}\right), \quad \sin^2(x), \quad x^x, \quad \frac{1}{1+\cos(x)}, \quad \arcsin(x^2+x+1)$$

We'd like to develop a process to integrate any rational function. We can do so, and it relies on two (to three) facts:

[FACT 1: Any polynomial can be factored into linear and quadratic factors

[EXAMPLES: $x^2+2x+1 = (x+1)^2$, $x^3 - x = (x+1)(x-1)\cdot x$, $x^3-3x^2+5x+14 = (x-2)^2(x+1)$

FACT 2: Any proper rational function (proper means numerator has lower degree than denominator) can be decomposed as a sum of proper rational functions of the form

$$\frac{A}{(\text{linear})^k} \quad \text{and} \quad \frac{Bx+C}{(\text{quadratic})^k}, \quad \text{where the "linear" and "quadratic"}$$

terms come from factorizing the original denominator, and "k" goes from 1 to the exponent in the original denominator. This is called the partial fraction decomposition.

This fact is much more complicated, so we'll understand it with increasingly complicated examples.

EXAMPLES: Find the partial fraction decompositions below:

① $\frac{1}{x^2+x}$

Fact 2 tells us to factor the denominator: $x^2+x = x(x+1)$. Then it claims we can write

$$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} \quad \text{for all } x.$$ Our job is to find A & B.

Method 1: Common denominators

Recombining the right side gives $\frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)}{x(x+1)} + \frac{Bx}{x(x+1)} = \frac{Ax+A+Bx}{x(x+1)}$

So we want $\frac{1}{x^2+x} = \frac{Ax+A+Bx}{x^2+x}$ $\rightsquigarrow$ $1 = Ax+A+Bx = (A+B)x+A$. We want these to be equal for any x, so the coefficients have to match: $0x+1 = (A+B)x+A$

so $0 = A+B$ and $1 = A$ which gives $B = -A = -(1) = -1$.

So, we get $\boxed{\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}}$ by plugging back in above.

Method 2: Plugging in.

Since we want $\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$, multiplying by x^2+x gives

$$1 = \frac{A(x^2+x)}{x} + \frac{B(x^2+x)}{x+1} \longrightarrow 1 = A(x+1) + Bx$$

To find A & B, we can plug in our favorite values of x. In this case, $x=0$ seems nice: $1 = A((0)+1) + B(0) = A$ so $A = 1$. We also see $x=-1$ is nice: $1 = A((-1)+1) + B(-1) = -B$ so $B = -1$. This is the same conclusion:

$$\boxed{\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}}$$

(2) $\frac{x+12}{x^2-x-6}$

Again, factorize $x^2-x-6 = (x-3)(x+2)$ so write $\frac{x+12}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$. Multiply by the denominator to get $x+12 = A(x+2) + B(x-3)$. Plugging in $x=-2$ gives $10 = B(-5)$ so $B = -2$ and when $x=3$: $15 = A(5)$ so $A = 3$.

So $\boxed{\frac{x+12}{x^2-x-6} = \frac{3}{x-3} - \frac{2}{x+2}}$

(3) $\frac{x}{(x-3)^2}$

This is our first example with an exponent. Fact 2 says we can write it in the form

$$\frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

for some constants A, B. Clearing denominators gives $x = A(x-3) + B$ and plugging in $x=3$ gives $3 = A(3-3) + B \longrightarrow B = 3$. This time, there isn't a second "cancelling" x-value, so we just plug in something arbitrary:

$x=0 \longrightarrow 0 = A(0-3) + (3) \rightarrow -3 = -3A \longrightarrow A = 1$. So,

$$\boxed{\frac{x}{(x-3)^2} = \frac{1}{x-3} + \frac{3}{(x-3)^2}}$$

(4) $\frac{4x-1}{x^3+x}$

Factoring gives $x^3+x = x(x^2+1)$, but x^2+1 cannot be factored further (x^2+1 is always positive, so it has no roots). So, we need the "quadratic" part of Fact 2:

$$\frac{4x-1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$ so by clearing denominators:

$4x-1 = A(x^2+1) + (Bx+C)x$. Plugging in $x=0$ gives $-1 = A(0^2+1)$

so $A = -1$. Rather than plugging in more values, let's now compare coefficients:

$4x-1 = -(x^2+1) + Bx^2 + Cx \longrightarrow Bx^2 + Cx = x^2 + 4x$ so $B=1$, $C=4$.

Thus, we conclude $\boxed{\dfrac{4x-1}{x^3+x} = \dfrac{x+4}{x^2+1} - \dfrac{1}{x}}$

⑤ $\dfrac{4x^3+2x+7}{(x^2+2x+2)^2(x-1)^3(x+1)x}$

We won't do this problem by hand, but the setup is that:

$$\frac{4x^3+2x+7}{(x^2+2x+2)^2(x-1)^3(x+1)x} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^2} + \frac{G}{(x-1)^3} + \frac{H}{x+1} + \frac{I}{x}$$

We could then proceed as usual to clear denominators and solve for A...I. The important part of this problem is to recognize the <u>form</u> of the final answer: how the factorization, exponents, and quadratic factors affect this form.

2 Section 1 was a lot of algebra, but we can now use it to integrate proper rational functions.

<u>EXAMPLES</u>: Use partial fraction decomposition to evaluate each integral.

① $\int \frac{1}{x^2+x}\,dx$

This was example ① from before:

$$\int \frac{1}{x^2+x}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + C$$

We can see how each partial fraction is easier to integrate than the original fraction, which needed a trig-sub, after completing the square.

② $\int \frac{x+12}{x^2-x-6}\,dx$

This was ② above: $\int \frac{x+12}{x^2-x-6}\,dx = \int \left(\frac{3}{x-3} - \frac{2}{x+2}\right)dx = 3\ln|x-3| - 2\ln|x+2| + C.$

③ $\int \frac{x}{(x-3)^2}\,dx$

This was ③: $\int \frac{x}{(x-3)^2}\,dx = \int \left(\frac{1}{x-3} + \frac{3}{(x-3)^2}\right)dx = \ln|x-3| - \frac{3}{x-3} + C$

[Recall $\int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{u} + C$].

Notice that this problem was also do-able via the substitution $u = x-3$, $du = dx$, so:

$$\int \frac{x}{(x-3)^2}\,dx = \int \frac{(u+3)}{u^2}\,du = \int \left(\frac{u}{u^2} + \frac{3}{u^2}\right)du = \int \left(\frac{1}{u} + \frac{3}{u^2}\right)du$$

$$= \ln|u| - \frac{3}{u} + C = \ln|x-3| - \frac{3}{x-3} + C.$$

In other words, partial fractions work, but use the method that seems easiest.

④ $\int \frac{4x-1}{x^3+x}\,dx$

This was ④: $\int \frac{4x-1}{x^3+x}\,dx = \int \left(\frac{x+4}{x^2+1} - \frac{1}{x}\right)dx.$

We can do the second one ($\int \frac{1}{x}\,dx$), but how do we handle $\int \frac{x+4}{x^2+1}\,dx$?

There are two ideas here: u-sub and trig-sub. Since u-sub is easier, we'll try it first: $u = x^2+1$ gives $du = 2x\,dx$, which handles the "x" in the numerator, but not the "1." But the "1" can be handled by trig sub. All together, we get:

$$\int \frac{4x-1}{x^3+x}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \int \frac{1}{x}\,dx$$

$$= \int \frac{x}{u}\cdot\frac{du}{2x} + \arctan(x) - \ln|x| + C$$

$$= \frac{1}{2}\int \frac{du}{u} + \arctan(x) - \ln|x| + C$$

$$= \frac{1}{2}\ln|u| + \arctan(x) - \ln|x| + C$$

$$= \frac{1}{2}\ln(x^2+1) + \arctan(x) - \ln|x| + C.$$

3 Finally, not every rational function is proper; some have a numerator of higher degree. We handle these as before: polynomial long division splits it into a polynomial plus a proper rational function.

EXAMPLE: Compute $\int \frac{2x^3+5x^2+6x+1}{x^2+x}\,dx$

This is improper, so we do polynomial division to write

$$\frac{2x^3+5x^2+6x+1}{x^2+x} = 2x+3+\frac{3x+1}{x^2+x}$$

$$\begin{array}{r} 2x+3 \\ x^2+x \;\overline{)\; 2x^3+5x^2+6x+1} \\ \underline{2x^3+2x^2} \\ 3x^2+6x+1 \\ \underline{3x^2+3x} \\ 3x+1 \end{array}$$

The remainder is proper so we factor $x^2+x = x(x+1)$ and write

$$\frac{3x+1}{x^2+x} = \frac{A}{x}+\frac{B}{x+1} \longrightarrow 3x+1 = A(x+1)+Bx$$ so we get for

$x=0$: $3(0)+1 = A(0+1)+B(0) \longrightarrow A = 1$

$x=-1$: $3(-1)+1 = A(-1+1)+B(-1) \longrightarrow -B = -2 \longrightarrow B = 2$.

Now we put it all together and integrate:

$$\int \frac{2x^3+5x^2+6x+1}{x^2+x}\,dx = \int \left(2x+3+\frac{1}{x}+\frac{2}{x+1}\right)dx = x^2+3x+\ln|x|+2\ln|x+1|+C$$

In summary, to integrate a rational function:

1. Do polynomial long division if needed.
2. Factorize the denominator into (possibly repeated) linear and irreducible quadratics.
3. Find the partial fraction decomposition of the proper fraction.
4. Reassemble all the terms and integrate:
   - (i) For polynomial terms, use the power rule
   - (ii) For linear denominators, use ln or power rule (negative) if there's an exponent.
   - (iii) For quadratics, split into a u-sub and a trig-sub.