§7.6

MAIN CONCEPTS

1 Special functions

2 Chapter summary.

1 This chapter involved many techniques for solving integrals. The unfortunate truth is that despite appearances, most integrals can't be solved explicitly. One of the main tools to handle this is numerical integration – using a computer to integrate – which is beyond the scope of this course.

Another tool is simply naming. If many different mathematicians are evaluating similar integrals, giving it a special name or symbol is useful for multiple reasons: (1) they can sometimes show that unrelated parts of math actually are connected, and (2) if a new technique arises for that specific integral, it helps multiple people at once. Here are some named functions, sometimes called "special functions":

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\,dt \quad , \quad S(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right)dt \quad , \quad Li(x) = \int_2^x \frac{\ln(t)}{t}\,dt$$

We can use these to solve some integrals, including substitutions and other transformations.

EXAMPLES: Evaluate each integral in terms of the given special functions.

① $\int \sin(x^2)\,dx$ in terms of $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$.

Use a substitution to make our integral look like the given one. Try $x = \sqrt{\frac{\pi}{2}}\cdot u$, so $dx = \sqrt{\frac{\pi}{2}}\,du$

and: $\int \sin(x^2)\,dx = \int \sin\left(\frac{\pi}{2}u^2\right)\cdot\sqrt{\frac{\pi}{2}}\,du = \sqrt{\frac{\pi}{2}}\int \sin\left(\frac{\pi u^2}{2}\right)du = \sqrt{\frac{\pi}{2}}S(u) + C = \sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{2}{\pi}}x\right) + C$

Notice that we've used both fundamental theorems here: $S'(u) = \sin\left(\frac{\pi u^2}{2}\right)$ by FTC1, and so $\int S'(u)\,du = S(u) + C$ by FTC2.

② $\int (2x+3)e^{-x^2}\,dx$ in terms of $E(x) = \int_0^x e^{-t^2}\,dt$.

Before this section, we would have tried a u-sub, and this is still useful; for $u = -x^2$, $du = -2x\,dx$, so:

$$\int (2x+3)e^{-x^2}\,dx = \int 2xe^{-x^2}\,dx + \int 3e^{-x^2}\,dx = -\int e^u\,du + 3\int e^{-x^2}\,dx = -e^u + 3E(x) + C$$

$$= -e^{-x^2} + 3E(x) + C$$

2 Overall, we've learned many techniques to evaluate integrals; the hard part, at this point, is knowing which one to use when. We try to summarize all of the techniques in the following flowchart:

