§8.1

MAIN CONCEPT

L'Hôpital's rule (0/0 form)

In this chapter, we'd like to find areas of "infinite" (unbounded) regions. To do this, we'll write the region as the limit of "finite" (bounded) regions and take a limit. To set up for this, we add a tool to our arsenal for evaluating limits.

L'Hôpital's Rule (0/0 form): If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ for some finite L or $L = \pm\infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ as well.

In other words, if evaluating a limit gives "$\frac{0}{0}$" but the derivatives give a limit, it gives the same answer.

EXAMPLES: Evaluate each limit.

① $\lim_{x\to 3} \frac{x^2-9}{x-3}$

We were able to solve limits like this before by factoring. Now, we try:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} \overset{\text{"="}}{=} \frac{(3)^2-9}{(3)-3} = \frac{9-9}{3-3} = \frac{0}{0}, \text{ so L.H.'s rule applies. Then,}$$

$$\lim_{x\to 3} \frac{x^2-9}{x-3} \overset{\text{L.H.}}{=} \lim_{x\to 3} \frac{2x}{1} = 2(3) = \boxed{6}.$$

② $\lim_{x\to 2} \frac{x^2-2x}{x^2-3x+2}$

Again, plug in $x=2$ to get $\frac{(2)^2-2(2)}{(2)^2-3(2)+2} = \frac{4-4}{4-6+2} = \frac{0}{0}$ so L.H. applies, and

$$\lim_{x\to 2} \frac{x^2-2x}{x^2-3x+2} \overset{\text{L.H}}{=} \lim_{x\to 2} \frac{2x-2}{2x-3} = \frac{2(2)-2}{2(2)-3} = \frac{4-2}{4-3} = \boxed{2}.$$

③ $\lim_{x\to 0} \frac{\sin(x)}{x}$

For $x=0$ we get $\frac{\sin(0)}{(0)} = \frac{0}{0}$ so $\lim_{x\to 0} \frac{\sin(x)}{x} \overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = \boxed{1}$.

④ $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

For $x=0$ we get $\frac{\sin(0)-(0)}{(0)^3} = \frac{0}{0}$, so $\lim_{x\to 0} \frac{\sin(x)-x}{x^3} \overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{\cos(x)-1}{3x^2} \overset{\text{"="}}{=} \frac{\cos(0)-1}{3(0)^2} = \frac{0}{0}$

So, we can use L.H. again: $\overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{-\sin(x)}{6x} \overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{-\cos(x)}{6} = \boxed{-\frac{1}{6}}$

($\frac{-\sin(0)}{6(0)} = \frac{0}{0}$)

⑤ $\lim_{x\to 0} \frac{\cos(x)}{x}$

Plugging in $x=0$ gives $\frac{\cos(0)}{(0)} = \frac{1}{0}$ so this diverges (DNE). A common mistake is to "apply L.H." here, but it <u>does not apply</u>.

For a brief explanation of why it works, we can do linear approximation of $f(x)$ and $g(x)$. We know that "near $x=a$" we have $f(x) \approx f(a) + f'(a)(x-a)$ [linear approximation] and similarly $g(x) \approx g(a) + g'(a)(x-a)$. By assumption, $\lim_{x\to a} f(x) = 0$, so $f(a) = 0$ and again similarly, $g(a) = 0$. Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{0 + f'(a)(x-a)}{0 + g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)} \quad \text{as claimed.}$$