§8.2

MAIN CONCEPTS

1. L'Hôpital's Rule ($\frac{\infty}{\infty}$ form)
2. Other indeterminate forms

---

1. Just like limits that give $\frac{0}{0}$, some limits give $\frac{\infty}{\infty}$ and also have a L'Hôpital's Rule:

L'HÔPITAL'S RULE ($\frac{\infty}{\infty}$ FORM): If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ exists or has $L = \pm\infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ as well.

EXAMPLES: Evaluate each limit.

① $\lim_{x\to\infty} \frac{x}{e^x}$

This evaluates to $\frac{\infty}{e^\infty} = \frac{\infty}{\infty}$ so L.H. applies, giving $\lim_{x\to\infty} \frac{x}{e^x} \overset{L.H.}{=} \lim_{x\to\infty} \frac{1}{e^x} = \left(\frac{1}{\infty}\right) = 0$.

② $\lim_{x\to\infty} \frac{x^3+x^2+3x}{e^x}$

Same idea: $\lim_{x\to\infty} \frac{x^3+x^2+3x}{e^x} \overset{L.H.}{=} \lim_{x\to\infty} \frac{3x^2+2x+3}{e^x} \overset{L.H.}{=} \lim_{x\to\infty} \frac{6x+2}{e^x} \overset{L.H.}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$

($\frac{\infty}{\infty}$, so applies; $\frac{\infty}{\infty}$; $\frac{\infty}{\infty}$; $\frac{6}{\infty}$)

These examples demonstrate an important fact: this fraction is zero because the denominator is much bigger than the numerator. In general, an <u>exponential in x is much larger than a power of x</u>, for x large.

③ $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$

Notice "$\ln(\infty) = \infty$" which we can see either from the graph or the fact that it's the inverse of e^x.

So, we do have $\frac{\infty}{\infty}$ and can use L.H.: $\lim_{x\to\infty} \frac{\ln(x)}{x^2} \overset{L.H.}{=} \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$.

($\frac{1}{\infty}$)

④ $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$

Same idea: $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} \overset{L.H.}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$.

($\frac{\infty}{\infty}$; $\frac{2}{\infty}$)

Again, we can extrapolate to a general fact: <u>ln(x) grows more slowly than any positive power of x</u>, for x large (even fractional powers).

⑤ $\lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\pi-2x)}{\tan(x)}$.

If we try to plug in $x = \frac{\pi}{2}$, we get $\ln(\pi - 2(\frac{\pi}{2})) = \ln(\pi-\pi) = \ln(0) = -\infty$ on top and $\tan(\frac{\pi}{2}) = \frac{\sin(\frac{\pi}{2})}{\cos(\frac{\pi}{2})} = \frac{1}{0}$ "=" ∞ on the bottom. So this is a valid situation to use L.H.:

$\lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\pi-2x)}{\tan(x)} \overset{L.H.}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\left(\frac{-2}{\pi-2x}\right)}{\sec^2(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{-2\cos^2(x)}{\pi-2x} \overset{L.H.}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{4\cos(x)\sin(x)}{-2} = 0$

($\frac{0}{0}$)

Notice that along the way, we got a $\frac{0}{0}$ form, so used L.H. like we did in §8.1.

2. The reason that $\frac{0}{0}$ and $\frac{\infty}{\infty}$ might exist is that there's a "balancing act" going on: dividing by something small (0) makes a big number while multiplying by it makes it small. The limit exists when one of these "wins" — pulling it to zero,

infinity, or somewhere between.

There are other forms that involve "balancing acts" — we call these indeterminate. The goal to evaluating such a limit is to convert it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form so we can use L.H. The most common indeterminate forms are $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, and 1^∞.

$0\cdot\infty$ form Rewriting this as $\frac{\infty}{1/0}$ or $\frac{0}{1/\infty}$ makes this in the desired form.

EXAMPLE: $\lim_{x\to 0^+} x\ln(x)$.

This approaches $(0)(-\infty)$ so it's the described type, and we can write:

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \overset{\frac{-\infty}{\infty}}{\overset{\text{L.H.}}{=}} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

$\infty-\infty$ form We try to combine this into a single term (common denominator or rationalize).

EXAMPLES: Compute each limit

① $\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right)$

These are both $\frac{1}{0^+} = \infty$, so we have the desired form. Making a common denominator:

$$\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right) = \lim_{x\to 1^+} \frac{x(\ln(x)) - (x-1)}{(x-1)\ln(x)} \overset{\frac{0}{0}}{\overset{\text{L.H.}}{=}} \lim_{x\to 1^+} \frac{\ln(x) + x(\frac{1}{x}) - 1}{\ln(x) + (x-1)(\frac{1}{x})} = \lim_{x\to 1^+} \frac{\ln(x)}{\ln(x) + 1 - \frac{1}{x}}$$

$$\overset{\frac{0}{0}}{\overset{\text{L.H.}}{=}} \lim_{x\to 1^+} \frac{1/x}{1/x + 1/x^2} = \frac{1}{1+1} = \frac{1}{2}.$$

② $\lim_{x\to\infty}\left(\sqrt{x+1} - \sqrt{x}\right)$

Multiplying by the conjugate $(\sqrt{x+1} + \sqrt{x})$ works here:

$$\lim_{x\to\infty}\left(\sqrt{x+1} - \sqrt{x}\right) = \lim_{x\to\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} = \lim_{x\to\infty} \frac{(x+1) - (x)}{\sqrt{x+1} + \sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0.$$

0^0, ∞^0, and 1^∞ forms These all involve exponents, so taking ln can help.

EXAMPLES: Compute each limit.

① $\lim_{x\to 0^+} (x+1)^{\cot(x)}$

This is of the form 1^∞. So, write $L = \lim_{x\to 0^+} (x+1)^{\cot(x)}$ and take ln:

$$\ln(L) = \lim_{x\to 0^+} \ln(x+1)^{\cot(x)} = \lim_{x\to 0^+} \cot(x)\cdot\ln(x+1) = \lim_{x\to 0^+} \frac{\ln(x+1)}{\tan(x)} \overset{\text{L.H.}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x+1}}{\sec^2(x)} = \frac{\frac{1}{0+1}}{\sec^2(0)} = 1.$$

$\frac{0}{0}$ form

Then we exponentiate to solve for $L = e^{\ln(L)} = e^1 = \boxed{e.}$

② $\lim_{x\to 0^+} (\sin(x))^x$

This is of the form 0^0. Write $L = \lim_{x\to 0^+} (\sin(x))^x$ so that

$$\ln(L) = \lim_{x\to 0^+} \ln(\sin(x)^x) = \underbrace{\lim_{x\to 0^+} x\ln(\sin(x))}_{(0)(-\infty) \text{ so } \to \frac{-\infty}{1/0}} = \lim_{x\to 0^+} \frac{\ln(\sin(x))}{1/x} \overset{\frac{-\infty}{\infty}}{\overset{\text{L.H.}}{=}} \lim_{x\to 0^+} \frac{\frac{1}{\sin(x)}\cdot\cos(x)}{-1/x^2}$$

$$= \lim_{x\to 0^+} -\frac{x^2\cos(x)}{\sin(x)} \overset{\frac{0}{0}}{\overset{\text{L.H.}}{=}} \lim_{x\to 0^+} -\frac{2x\cos(x) - x^2\sin(x)}{\cos(x)} = \frac{2(0)\cos(0) - (0)^2\sin(0)}{\cos(0)} = \frac{0}{1} = 0.$$

So $\ln(L) = 0$ and $L = e^0 = 1$.