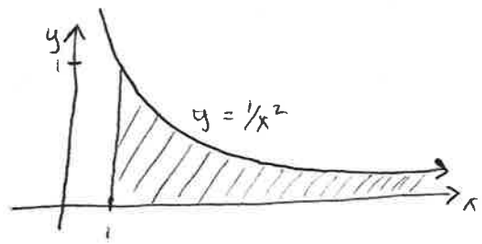
§8.3

MAIN CONCEPTS

1. Infinite bound of integration.
2. Both bounds infinite
3. (optional) Gabriel's Horn.

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1 As suggested at the beginning of the chapter, we've established rules for limits so we can deal with new "infinite objects" — namely, the area of an unbounded region. Consider, for example, the graph of $y = \frac{1}{x^2}$. We know how to find the area under this curve between two vertical lines by taking an integral. But what about all of the area to the right of $x=1$ (as shown)? We could integrate from $x=1$ to $x=100$ and get Area $= \int_1^{100} \frac{1}{x^2}\,dx$ or go even further to $\int_1^{1000} \frac{1}{x^2}\,dx$ or $\int_1^{1000000} \frac{1}{x^2}\,dx$. We can see that what we really want is



$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(\frac{x^{-1}}{-1}\Big|_1^b \right) = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1} \right) = 1 - \lim_{b\to\infty} \frac{1}{b} = 1 - 0 = 1.$$

In other words, we conclude that there is exactly 1 unit of shaded area above.

DEFINITION: We'll write $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$ for any function $f(x)$ integrable on $[a,\infty)$. Similarly, define $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$. These are called improper integrals. If this limit exists, we say that the improper integral converges; otherwise, we say it diverges.

EXAMPLES: Compute each improper integral, or show that it diverges.

① $\int_0^\infty \frac{1}{1+x^2}\,dx$

By definition, $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \left[\arctan(x)\Big|_0^b \right] = \lim_{b\to\infty} \arctan(b) - \arctan(0)$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$. So it converges to $\pi/2$.

② $\int_5^\infty \frac{1}{x}\,dx$

By definition, $\int_5^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_5^b \frac{1}{x}\,dx = \lim_{b\to\infty} \left[\ln(x)\Big|_5^b \right] = \lim_{b\to\infty} \ln(b) - \ln(5) = \infty$

since $\lim_{b\to\infty} \ln(b) = \infty$. So this diverges.

③ $\int_{-\infty}^3 xe^{-x^2}\,dx$ =

Same idea: $\int_{-\infty}^3 xe^{-x^2}\,dx = \lim_{a\to-\infty} \int_a^3 xe^{-x^2}\,dx = \lim_{a\to-\infty} \int_{-a^2}^{-9} xe^u \cdot \frac{du}{-2x} = -\frac{1}{2} \lim_{a\to-\infty} \int_{-a^2}^{-9} e^u\,du$

$u = -x^2$

$du = -2x\,dx$

$= -\frac{1}{2}\lim_{a\to-\infty}\left[e^u\Big|_{-a^2}^{-9}\right] = -\frac{1}{2}\lim_{a\to-\infty}\left(e^{-9}-e^{-a^2}\right) = -\frac{1}{2}e^{-9} + \frac{1}{2}\lim_{a\to-\infty}\frac{1}{e^{a^2}} = -\frac{1}{2}e^{-9}$

since $\frac{1}{e^{a^2}} \to 0$ as $a\to-\infty$. So it converges to $-\frac{1}{2}e^{-9}$.

Among these, ② has special significance and is worth generalizing. We saw in the introduction that $\int_1^\infty \frac{1}{x^2}dx$ converges while example ② shows $\int_1^\infty \frac{1}{x}dx$ diverges. What about other exponents?

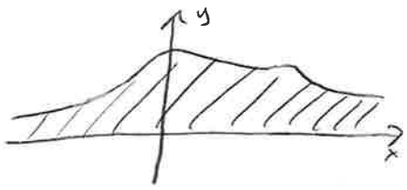
EXAMPLE: Find all values of p for which $\int_1^\infty \frac{1}{x^p}dx$ converges.

We already know it doesn't converge for $p=1$, so now let's think about $p\neq 1$. Then, we integrate:

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b x^{-p}dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\Big|_1^b\right] = \lim_{b\to\infty}\frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}.$$

This limit converges only if the exponent is negative, because then the "∞" is in the denominator. So, it converges if $1-p<0$, i.e. $1<p$. So $\int_1^\infty \frac{1}{x^p}dx$ converges $\iff p>1$.

2 We can also combine these ideas to integrate a function on all x-values to get the total area under a curve in both directions.



DEFINITION: We'll write $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$, and again say this converges if this value exists and diverges otherwise.

EXAMPLES: Compute each improper integral or show that it diverges.

① $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$

Following the definition gives:

$$\int_{-\infty}^\infty \frac{1}{1+x^2}dx = \int_{-\infty}^0 \frac{1}{1+x^2}dx + \int_0^\infty \frac{1}{1+x^2}dx = \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}dx$$

$$= \lim_{a\to-\infty}\left[\arctan(x)\Big|_a^0\right] + \lim_{b\to\infty}\left[\arctan(x)\Big|_0^b\right] = \arctan(0) - \lim_{a\to-\infty}\arctan(a) + \lim_{b\to\infty}\arctan(b) - \arctan(0)$$

$$= -\left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) = \pi.$$

So this integral converges to π.

② $\int_{-\infty}^\infty \frac{2x}{1+x^2}dx$

We'll use the substitution $u = 1+x^2$, so $du = 2x\,dx$, whence $\int \frac{2x}{1+x^2}dx = \int \frac{du}{u} = \ln|u| + C = \ln(1+x^2) + C$.

For the improper (definite) integrals, we thus get:

$$\int_{-\infty}^\infty \frac{2x}{1+x^2}dx = \int_{-\infty}^0 \frac{2x}{1+x^2}dx + \int_0^\infty \frac{2x}{1+x^2}dx = \lim_{a\to-\infty}\left[\ln(1+x^2)\Big|_a^0\right] + \lim_{b\to\infty}\left[\ln(1+x^2)\Big|_0^b\right]$$

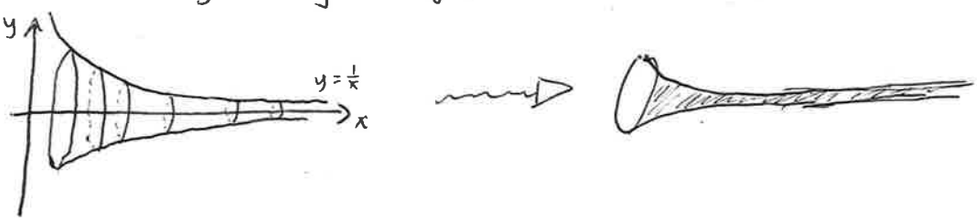
$$= \lim_{a\to\infty}\left(\ln(1+0^2) - \ln(1+a^2)\right) + \lim_{b\to\infty}\left(\ln(1+b^2) - \ln(1+0^2)\right)$$

$$= -\lim_{a\to-\infty}\ln(1+a^2) + \lim_{b\to\infty}\ln(1+b^2) \qquad \text{DNE}.$$

These limits do not exist, so the integral diverges.

Note: we need to be careful here. It looks like the two terms cancel, since the first is "$-\ln(1+(-\infty)^2) = -\ln(1+\infty^2)$" and the second is "$\ln(1+\infty^2)$". But the two limits need to be taken separately and then subtracted. So unlike §8.2, there's no chance to get an "$\infty - \infty$" cancellation.

3 This last section is an amusing paradox, which often arise when we apply (finite) intuition to infinite objects. Consider a "trumpet" formed by rotating the graph of $y = \frac{1}{x}$, $x \geq 1$ around the x-axis.



This object is sometimes called Gabriel's Horn, or Torricelli's Trumpet.

We remember, from a previous class (§5.2), how to find the horn's volume:

$$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2}\,dx$$ which converges (to π) since $p > 1$.

(see the last example in section 1).

On the other hand, we can also find its surface area (§5.4):

$$A = \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left[\frac{d}{dx}\left(\frac{1}{x}\right)\right]^2}\,dx = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^2}}\,dx.$$

But $\frac{1}{x^2} \geq 0$, so $1+\frac{1}{x^2} \geq 1$ and $\sqrt{1+\frac{1}{x^2}} \geq \sqrt{1} = 1$, so we get:

$$A = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^2}}\,dx \geq 2\pi\int_1^\infty \frac{1}{x}(1)\,dx = 2\pi\int_1^\infty \frac{1}{x}\,dx = \infty$$

since we saw that this diverges ($p=1$).

So, the horn has finite volume but infinite surface area! In other words, you ~~need a~~ could fill it with paint (just π units^3), but need an infinite amount of paint to cover it.