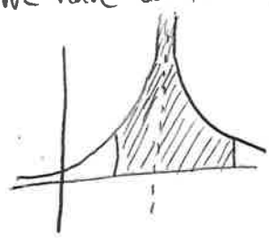
§8.4

MAIN CONCEPTS

1. Improper integrals : endpoint asymptote.
2. Improper integrals: interior asymptote.

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1. We have another type of infinite region whose area we can inquire about: those arising from a vertical asymptote. Again, we handle these by integrating on more reasonable regions and taking a limit.



DEFINITION: If $f(x)$ is integrable on $(a,b]$ then we define $\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$ and similarly define $\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$ if $f(x)$ is integrable on $[a,b)$.

The point is that $f(x)$ doesn't need to be "nice" (continuous, bounded, etc.) at one of the endpoints as long as we can approach it.

EXAMPLES: Compute each improper integral, or explain why it diverges.

① $\int_0^{16} \frac{1}{\sqrt[4]{x}}dx$

Notice that this is improper since $\frac{1}{\sqrt[4]{x}} = x^{-1/4}$ has an asymptote at $x=0$. So,

$$\int_0^{16}\frac{1}{\sqrt[4]{x}}dx = \lim_{a\to 0^+}\int_a^{16} x^{-1/4}dx = \lim_{a\to 0^+}\left[\frac{x^{3/4}}{3/4}\Big|_a^{16}\right] = \frac{4}{3}\lim_{a\to 0^+}\left[16^{3/4} - a^{3/4}\right] = \frac{4}{3}(8-0) = \boxed{\frac{32}{3}}$$

② $\int_0^2 \frac{1}{\sqrt{4-x^2}}dx$

Same idea; this is improper because of the asymptote at $x=2$. To evaluate this, use the trig substitution $x = 2\sin(u)$, so $dx = 2\cos(u)du$, and we have:  So:

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}dx = \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-x^2}}dx = \lim_{b\to 2^-}\int_{\cdot}^{\cdot}\frac{2\cos(u)}{\sqrt{4-4\sin^2(u)}}du = \lim_{b\to 2^-}\int_{\cdot}^{\cdot}\frac{\cos(u)}{\cos(u)}du = \lim_{b\to 2^-}\int du$$

$$= \lim_{b\to 2^-} u\Big|_{\cdot}^{\cdot} = \lim_{b\to 2^-}\left[\arcsin\left(\frac{x}{2}\right)\Big|_0^b\right] = \lim_{b\to 2^-}\left(\arcsin\left(\frac{b}{2}\right) - \arcsin(0)\right)$$

$$= \arcsin(1) - 0 = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

③ $\int_0^5 \frac{1}{x^2}dx$

Asymptote at $x=0$: $\int_0^5 \frac{1}{x^2}dx = \lim_{a\to 0^+}\int_a^5 x^{-2}dx = \lim_{a\to 0^+}\left[\frac{x^{-1}}{-1}\Big|_a^5\right] = \lim_{a\to 0^+}\left(\frac{1}{a} - \frac{1}{5}\right)$ DNE

Since the limit gives ∞ as the first term. So this integral **diverges.**

2. We can also handle asymptotes in the middle of an interval.

DEFINITION: If $f(x)$ is integrable ~~except~~ on $[a,b]$ except possibly at $x=c$, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

EXAMPLES: Evaluate each improper integral or, explain why it diverges.

① $\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\,dx$

This is improper because of the asymptote at $x=0$. So:

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\,dx = \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{1} x^{-1/3}\,dx = \lim_{b\to 0^-}\int_{-1}^{b} x^{-1/3}\,dx + \lim_{a\to 0^+}\int_{a}^{1} x^{-1/3}\,dx =$$

$$= \lim_{b\to 0^-}\left[\frac{x^{2/3}}{2/3}\Big|_{-1}^{b}\right] + \lim_{a\to 0^+}\left[\frac{x^{2/3}}{2/3}\Big|_{a}^{1}\right] = \frac{3}{2}\left[\lim_{b\to 0^-}\left(b^{2/3}\right) - (-1)^{2/3} + (1)^{2/3} - \lim_{a\to 0^+}\left(a^{2/3}\right)\right]$$

$$= \frac{3}{2}(0 - 1 + 1 - 0) = 0.$$

② $\int_{-2}^{2} \frac{1}{x^5}\,dx$

Same idea: $\int_{-2}^{2} \frac{1}{x^5}\,dx = \int_{-2}^{0} x^{-5}\,dx + \int_{0}^{2} x^{-5}\,dx = \lim_{b\to 0^-}\int_{-2}^{b} x^{-5}\,dx + \lim_{a\to 0^+}\int_{a}^{2} x^{-5}\,dx =$

$$= \lim_{b\to 0^-}\left[\frac{x^{-4}}{-4}\Big|_{-2}^{b}\right] + \lim_{a\to 0^+}\left[\frac{x^{-4}}{-4}\Big|_{a}^{2}\right] = -\frac{1}{4}\left[\lim_{b\to 0^-}\left(\frac{1}{b^4}\right) - \frac{1}{(-2)^4} + \frac{1}{2^4} - \lim_{a\to 0^+}\frac{1}{a^4}\right]$$

which doesn't exist ($\frac{1}{0}$ in two spots). So this integral diverges.