§9.1

MAIN CONCEPTS

1. Infinite sequences
2. Convergence.

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1. A sequence is a list of numbers, which for us is always infinite in the sense that there are always more available terms. Let's describe four ways to write a sequence.

(I) In words. For example, "the list of odd numbers" defines a sequence. Notice that this is somewhat ambiguous—Which term is first? Does this include negative odd numbers? Maybe better would be "the positive odd whole numbers in increasing order starting from 1."

(II) Pattern recognition. Often, if we think that a sequence is recognizable, we might list a few terms with an ellipsis: $1, 3, 5, 7, 9, \ldots$ or $1, 4, 9, 16, 25, \ldots$, or $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

(III) Explicit formula. The issue with both of the previous approaches is the lack of precision. An explicit formula addresses this by coming up with a rule to go from integers to terms in the sequence. This is written as $a_n = \cdots$ for $n \geq \bullet\bullet\bullet$. For example, the sequence $1, 3, 5, 7, \ldots$ could be written as $a_n = 2n+1$ for $n \geq 0$. This is telling us to plug in integers for "n" starting at 0. So $a_0 = 2(0)+1 = 1$, $a_1 = 2(1)+1 = 3$, $a_2 = 2(2)+1 = 5$, and $a_3 = 2(3)+1 = 7$. This is again giving us the odd numbers in order.

Notice that where "n" starts is a choice. We could also have written $a_n = 2n-1$ for $n \geq 1$. Then $a_1 = 2(1)-1 = 1$, $a_2 = 2(2)-1 = 3$, etc. giving the same sequence.

(IV) Recursive formula. Sometimes we can't write a formula for the n^{th} term like above, but we know how to go from one term to the next term. In this case we can write this "recursive" definition with the first term of the sequence as a base case. For example, $a_0 = 0$ and $a_{n+1} = 2a_n + 1$. Plugging in $n=0$ gives $a_1 = 2a_0 + 1 = 2(0)+1 = 1$. Plugging in $n=1$ gives $a_2 = 2a_1 + 1 = 2(1)+1 = 3$, and $a_3 = 2a_2 + 1 = 2(3)+1 = 7$.

2. The main question we have for sequences is whether the terms approach a specific value. This is asking what happens as n approaches ∞. Our main tool is that if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$ as well (the first statement is when x approaches infinity as a real number, including decimals, while the second is when n approaches infinity as an integer). In this case, we say the sequence converges to L.

EXAMPLES: Decide if each sequence converges, and if so, to what value.

① $a_n = \frac{3n^2}{7n^2+1}$ for $n \geq 0$.

We want to compute: $\lim_{n\to\infty} \frac{3n^2}{7n^2+1} = \lim_{n\to\infty} \frac{3}{7+1/n^2} = \frac{3}{7+0} = \frac{3}{7}$.

So this converges.

(2) $a_1 = 1$, $a_{n+1} = \dfrac{a_n}{a_n + 1}$.

It's not clear what the limit is since we don't have an explicit formula, so it might be useful to try and find an explicit one. We'll do a few terms; $a_2 = \dfrac{a_1}{a_1+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$, $a_3 = \dfrac{a_2}{a_2+1} = \dfrac{1/2}{1/2+1} \cdot \dfrac{2}{2} = \dfrac{1}{1+2} = \dfrac{1}{3}$, $a_4 = \dfrac{a_3}{a_3+1} = \dfrac{1/3}{1/3+1} = \dfrac{1/3}{4/3} = \dfrac{1}{4}$. We seem to have $a_n = \frac{1}{n}$, and indeed if $a_n = \frac{1}{n}$ then $a_{n+1} = \dfrac{1/n}{1/n+1} \cdot \dfrac{n}{n} = \dfrac{1}{1+n} = \dfrac{1}{n+1}$ ✓

So we found the explicit formula $a_n = \frac{1}{n}$ whence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

(3) $a_n = \sin(\pi n)$, for $n \geq 1$.

We have an explicit formula, but $\lim_{x\to\infty} \sin(\pi x)$ doesn't exist since sine oscillates. But remember that n is an integer, so we have $a_1 = \sin(\pi) = 0$, $a_2 = \sin(2\pi) = 0$, $a_3 = \sin(3\pi) = 0$, ... that is, $a_n = 0$ for all $n \geq 1$. So $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$.

(4) $a_n = \cos(\pi n)$ for $n \geq 0$.

We know better now and plug in n values: $a_0 = \cos(0) = 1$, $a_1 = \cos(\pi) = -1$, $a_2 = \cos(2\pi) = 0$, $a_3 = \cos(3\pi) = -1$, $a_4 = \cos(4\pi) = 1$, etc. So $a_n = (-1)^n$ flips between $+1$ and -1, and so doesn't approach any value. I.e. a_n diverges.