§9.2

MAIN CONCEPTS

1. Infinite series
2. Geometric series
3. Telescoping series
4. Test for divergence

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1 Suppose we tried to add infinitely many numbers, e.g.

$$0.9 + 0.09 + 0.009 + 0.0009 + 0.00009 + \cdots$$

We expect the sum to be $0.999999\ldots = 1$, insofar as "adding" infinitely many things even makes sense. Let's make this more formal.

Start with a sequence $a_0, a_1, a_2, \ldots$. Then we want to define $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots$. The notation "$\sum_{n=a}^{b} y$" says "plug in $n=a$, $n=a+1$, $n=a+2$, ..., $n=b-1$, $n=b$ into y and add up the results." We can't add infinitely many things, but we could add 10 or 100 or 1000 of them to get $\sum_{n=0}^{10} a_n$ or $\sum_{n=0}^{100} a_n$ or $\sum_{n=0}^{1000} a_n$. This gives a new sequence of <u>partial sums</u>: $S_N = \sum_{n=0}^{N} a_n$.

If this new sequence converges to a value S, we say that $\sum_{n=0}^{\infty} a_n = S$. I.e:

[<u>DEFINITION</u>: $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$, I.e. $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$

<u>EXAMPLE</u>: Add the above series $(0.9+0.09+\cdots)$.

We have $a_n = \frac{9}{10^n}$ for $n \geq 1$. The partial sums are $S_N = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \frac{9}{10^n}$. Let's calculate them: $S_1 = a_1 = 0.9$, $S_2 = a_1 + a_2 = 0.9 + 0.09 = 0.99$

$S_3 = a_1 + a_2 + a_3 = 0.9 + 0.09 + 0.009 = 0.999$, and

$S_4 = a_1 + a_2 + a_3 + a_4 = 0.9 + 0.09 + 0.009 + 0.0009 = 0.9999$.

We can rewrite them: $S_1 = 1 - \frac{1}{10}$, $S_2 = 1 - \frac{1}{100}$, $S_3 = 1 - \frac{1}{1000}$, and $S_4 = 1 - \frac{1}{10000}$.

I.e. we get $S_N = 1 - \frac{1}{10^N}$, so $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{10^N}\right) = 1 - \frac{1}{\infty} = 1 - 0 = \boxed{1}$.]

2 We see that this process is a bit tedious. Our first solution is to identify special sums that can be computed. Here's our first:

[<u>DEFINITION</u>: If $a \neq 0$ and r is any real number, then $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ar^4 + \cdots$ is called a <u>geometric series</u>, with initial term "a" and common ratio "r".

We can analyze this directly. First, we can write down a partial sum:

$S_N = a + ar + ar^2 + \cdots + ar^{N-1} + ar^N$. Multiplying this by r gives:

$rS_N = ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1}$.

If we subtract these two lines, most of it cancels, leaving:

$$S_N - rS_N = a - ar^{N+1} \longrightarrow S_N(1-r) = a - ar^{N+1} \longrightarrow S_N = \frac{a - ar^{N+1}}{1-r}.$$

Or, in words, $S_N = \dfrac{(\text{first term}) - (\text{next term})}{1 - (\text{ratio})}$.

EXAMPLE: Evaluate $1 + 10 + 100 + 1000 + \cdots + \underbrace{1000000000}_{=10^9}$ as a single term.

In fact, this isn't so bad: the sum is 1,111,111,111. But also we can see this as a geometric series since these are powers of 10. So, this sum is

$$\frac{(\text{first}) - (\text{next})}{1 - (\text{ratio})} = \frac{1 - 10^{10}}{1 - 10} = \frac{10^{10} - 1}{9}, \text{ which is the same value.}$$

We wanted to know when a series converges, so we should take $N \to \infty$ above. Then, $\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty}\left(\frac{a - ar^{N+1}}{1-r}\right)$. If $|r|<1$, then large powers of r give smaller numbers. So if $-1<r<1$, then this limit exists and equals $\frac{a-0}{1-r} = \frac{a}{1-r}$.

If $r \geq 1$ or $r \leq -1$, then the opposite happens, so the limit DNE. We've concluded

THEOREM: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $-1<r<1$, and diverges otherwise.

EXAMPLES: Evaluate each geometric series.

① $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

Each term is the previous times $\frac{1}{2}$. So this is geometric with $a = 1$ and $r = \frac{1}{2}$. So the sum is

$$\frac{a}{1-r} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{1/2} = \boxed{2}$$

② $0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

This is also geometric! The first term is $a = 0.9$ and each term is $\frac{1}{10}$ times the previous, so $r = \frac{1}{10}$. Then the sum is $\frac{0.9}{1 - 1/10} = \frac{9/10}{9/10} = \boxed{1}$. Again we've shown this sum to be 1.

③ $8 + 40 + 200 + 1000 + 5000 + 25000 + \cdots$

This is also geometric with $a = 8$ and $r = 5$. But $|r| \geq 1$, so this diverges.

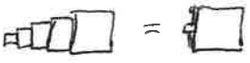
3 The advantage of a geometric series is that we can write the partial sums explicitly because of the cancellations. A second type is telescoping:

DEFINITIONS: A series of the form $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ is called a telescoping series.

Again, we can analyze directly:

$$S_N = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_{N-1} - a_N) + (a_N - a_{N+1})$$

$$= a_1 - a_{N+1}.$$

NOTE: This is why it's called telescoping — it collapses like a ~~pirate~~ sailor's telescope  =

Again, this is amenable to taking the limit directly to find the sum.

EXAMPLES: Evaluate each telescoping series.

① $\sum_{n=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)$

This is exactly the setup: $S_N=\sum_{n=1}^{N}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)=\left(\sin(1)-\sin\left(\frac{1}{2}\right)\right)+\left(\sin\left(\frac{1}{2}\right)-\sin\left(\frac{1}{3}\right)\right)+\cdots+\left(\sin\left(\frac{1}{N}\right)-\sin\left(\frac{1}{N+1}\right)\right)$

$=\sin(1)-\sin\left(\frac{1}{N}\right)$. Thus, the sum is

$\lim_{N\to\infty} S_N=\lim_{N\to\infty}\left(\sin(1)-\sin\left(\frac{1}{N}\right)\right)=\sin(1)-\sin(0)=\boxed{\sin(1)}$ and it converges.

② $\sum_{n=0}^{\infty}\left(\cos(n)-\cos(n+1)\right)$

Again, we get $S_N=\sum_{n=0}^{N}\left(\cos(n)-\cos(n+1)\right)=\left(\cos(0)-\cos(1)\right)+\left(\cos(1)-\cos(2)\right)+\cdots+\left(\cos(N)-\cos(N+1)\right)$

$=\cos(0)-\cos(N+1)$. Then $\lim_{N\to\infty}\left(\cos(0)-\cos(N+1)\right)$ DNE since cosine is oscillating. So this sum $\boxed{\text{diverges.}}$

③ $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

This doesn't look telescoping, but we can use partial fraction decomposition first: $\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$.

$\longrightarrow 1=A(n+1)+Bn$. Then $n=0\longrightarrow 1=A$

$n=-1\longrightarrow 1=-B\longrightarrow B=-1$. So $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$,

and $S_N=\sum_{n=1}^{N}\frac{1}{n(n+1)}=\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{1}-\frac{1}{N+1}$, so $\lim_{N\to\infty} S_N=\lim_{N\to\infty}\left(1-\frac{1}{N+1}\right)=1-0=\boxed{1.}$

④ A common theme in example ② above, and example ③ before that is that they diverge and the terms ~~converge~~ also diverge. This and more is true.

TEST FOR DIVERGENCE: If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n=0$. Equivalently, if $\lim_{n\to\infty} a_n\neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges.

EXAMPLES: Apply the test for divergence.

① $8+40+200+1000+5000+\cdots$

This is $\sum_{n=0}^{\infty} 8\cdot 5^n$, but $\lim_{n\to\infty} 8\cdot 5^n=\infty\neq 0$. So this diverges.

② $\sum_{n=0}^{\infty}(\cos(n)-1)$

Here, $\lim_{n\to\infty}(\cos(n)-1)$ DNE, so it $\neq 0$ and it diverges.

③ $\sum_{n=\phi}^{\infty}\frac{2n^2}{3n^2+n}$

Here, $\lim_{n\to\infty}\frac{2n^2}{3n^2+n}=\frac{2}{3}\neq 0$ so the sum diverges. This is clear to understand:

$S=\frac{2\cdot\phi^2}{3\cdot 1^2+1}+\frac{2\cdot 2^2}{3\cdot 2^2+2}+\frac{2\cdot 3^2}{3\cdot 3^2+3}+\cdots+\underbrace{\left(\sim\frac{2}{3}\right)+\left(\sim\frac{2}{3}\right)+\left(\sim\frac{2}{3}\right)+\cdots}$

This becomes infinite!

④ $\sum_{n=1}^{\infty}\frac{1}{n}$

Here, $\lim_{n\to\infty}\frac{1}{n}=0$. In this case, the test has <u>no conclusion</u>, and need to keep thinking.