§9.3

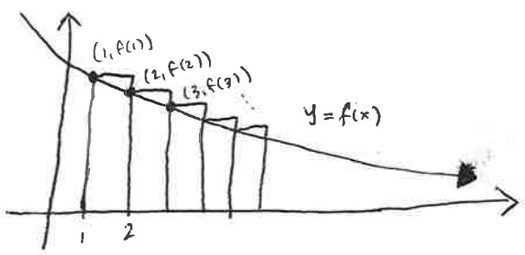
MAIN CONCEPTS

[1] Integral Test

[2] p-Series.

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[1] Evaluating a series is difficult. The goal of this and the next few sections is to determine if a series converges or diverges without necessarily evaluating it. Our first strategy will be to use geometry. Suppose $a_n = f(n)$ for some function $f(x)$. Graphing it gives:




In this picture, we've drawn rectangles, each of width = 1. The first has height $a_1$, the second has height $a_2$, the third has height $a_3$, etc.

The picture also shows that (Area of rectangles) $\geq$ (Area under curve). I.e. $a_1 + a_2 + a_3 + \cdots \geq \int_1^\infty f(x)\,dx$. If $\int_1^\infty f(x)\,dx = \infty$, then the series also has to be $\infty$, i.e. if the integral diverges, so does the series. In fact, by shifting the rectangles 1 unit left, we can reach the reverse conclusion, too. The only things we're missing are the technical details — e.g. that this picture only works if $f(x)$ is decreasing. Overall, we conclude:

INTEGRAL TEST: Let $a_n$ be a sequence with $a_n = f(n)$, where $f(x) > 0$ for all $x$ and $f(x)$ is decreasing. Then $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

NOTE: "If and only if" means that both statements imply each other. So if $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges, and vice versa.

EXAMPLES: Apply the integral test.

① $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$.

The function $f(x) = \frac{1}{1+x^2}$ is positive, and decreasing (if $a < b$, then $\frac{1}{1+a^2} > \frac{1}{1+b^2}$). So the test applies and

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \left[\arctan(x)\Big|_1^b\right]$$

$$= \lim_{b\to\infty} \arctan(b) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

This converged, so $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges too. NOTE: it isn't the case that $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ equals $\frac{\pi}{4}$, we just know that it converges to something.

② $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$.

Again, $f(x) = \frac{1}{x\ln(x)} \geq 0$ for $x \geq 2$ and is decreasing since the denominator is growing.

So, $\int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u}\,du = \lim_{b\to\infty} \left[ \ln(\ln(b)) \Big|_{\ln(2)}^{\ln(b)} \right]$

$u = \ln(x)$ $\quad x=2 \to u = \ln(2)$
$du = \frac{1}{x}dx$ $\quad x=b \to u = \ln(b)$

$= \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(2))$ DNE since $b\to\infty$ gives $\ln(b)\to\infty$ gives $\ln(\ln(b))\to\infty$.

So the integral diverges and so $\boxed{\sum_{n=2}^\infty \frac{1}{n\ln(n)} \text{ diverges}}$ too.

2. There's a class of series that occur often:

DEFINITION: A series of the form $\sum_{n=1}^\infty \frac{1}{n^p}$ for some constant $p$ is called a p-series.

When does a p-series converge? If $p \leq 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so it diverges by the test for divergence (§9.2). If $p > 0$, then $f(x) = \frac{1}{x^p}$ is positive and decreasing. Further, we know $\int_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p > 1$ (§8.3). So, we conclude:

THEOREM: A p-series converges if and only if $p > 1$.

EXAMPLES: Determine if each series converges or diverges.

① $\sum_{n=1}^\infty \frac{1}{n}$.

This is exactly the p-series with $p = 1$, so it diverges.

② $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$.

This is $p = 1/2$, so it diverges.

③ $\sum_{n=3}^\infty \frac{1}{n^2}$.

This is a p-series with $p = 2$ (the missing terms $\frac{1}{1^2} + \frac{1}{2^2}$ can't change convergence), so this converges.

④ $\sum_{n=1}^\infty \frac{4}{n^3}$

This is $\frac{4}{1^3} + \frac{4}{2^3} + \frac{4}{3^3} + \cdots = 4\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right) = 4\sum_{n=1}^\infty \frac{1}{n^3}$. This is 4 times a convergent p-series ($p=3$) so it converges.

⑤ $\sum_{n=1}^\infty \left(\frac{1}{n^{1.5}} + \frac{1}{n^5}\right)$

This is $\left(\frac{1}{1^{1.5}} + \frac{1}{1^5}\right) + \left(\frac{1}{2^{1.5}} + \frac{1}{2^5}\right) + \cdots = \left(\frac{1}{1^{1.5}} + \frac{1}{2^{1.5}} + \cdots\right) + \left(\frac{1}{1^5} + \frac{1}{2^5} + \cdots\right) = \sum_{n=1}^\infty \frac{1}{n^{1.5}} + \sum_{n=1}^\infty \frac{1}{n^5}$.

Each of these converges ($p=1.5$, $p=5$), so this sum converges.