§9.4

MAIN CONCEPTS

1. (Ordinary) Comparison Test
2. Limit Comparison Test
3. Ratio Test

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1. We learned last section that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. What about $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$? We know that convergence is a limit question, so really we need to understand $\frac{1}{n^3+1}$ when n is really big. But then $n^3 \approx n^3+1$ (e.g. if $n=100$, then $n^3 = 1{,}000{,}000$ and $n^3+1 = 1{,}000{,}001$ is barely different). So, we'd like to compare the two series: the one we understand ($\sum_{n=1}^{\infty} \frac{1}{n^3}$) and the one we'd like to understand ($\sum_{n=1}^{\infty} \frac{1}{n^3+1}$). This is possible:

(ORDINARY) COMPARISON TEST: Suppose $0 \le a_n \le b_n$ for all n.

i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, too.

ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges, too.

NOTE/MNEMONIC: What this test is saying is that if we have a smaller series (a_n) and a larger series (b_n), then if the bigger one is finite, the smaller is, too; and if the smaller one is infinite, the bigger one is, too.

EXAMPLES: Determine if each series converges or diverges.

① $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$.

As in the above discussion, we want to compare to $\frac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series, $p=3$), we want it to be the bigger one, i.e. we <u>want</u> $\frac{1}{n^3+1} \le \frac{1}{n^3}$ for all n. Is this true? Yes; the first has a bigger denominator, so it's smaller. So, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ indeed converges!

② $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Again, for large n, $\frac{1}{2n-1} \approx \frac{1}{2n}$. But $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p=1$), so we want it to be the smaller one, i.e. we want: $\frac{1}{2n-1} \ge \frac{1}{2n}$. Again, this is true ($2n-1 \le 2n \rightarrow$ bigger denominator), so $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges, too.

③ $\sum_{n=1}^{\infty} \frac{1}{n^2-5}$

For large n, $\frac{1}{n^2-5} \approx \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so we <u>want</u> $\frac{1}{n^2-5} \le \frac{1}{n^2}$. But this is <u>false</u>, since $n^2-5 \le n^2$. What should we do? The issue is our comparison series ($\frac{1}{n^2}$) wasn't big enough. So we can try replacing it with something bigger. Let's try $\frac{10}{n^2}$ instead.

Then $\sum_{n=1}^{\infty} \frac{10}{n^2} = 10\sum_{n=1}^{\infty} \frac{1}{n^2}$ still converges, so we'd want $\frac{1}{n^2-5} \overset{?}{\leq} \frac{10}{n^2}$. This is not so obvious, so we cross-multiply to get $n^2 \overset{?}{\leq} 10(n^2-5) = 10n^2 - 50$. $\longleftrightarrow$ $50 \leq 9n^2$. This is not always true, but is true for $n \geq 3$. So,

$$\sum_{n=1}^{\infty} \frac{1}{n^2-5} = \frac{1}{1^2-5} + \frac{1}{2^2-5} + \sum_{n=3}^{\infty} \frac{1}{n^2-5} \leq \frac{1}{-4} + \frac{1}{-1} + \sum_{n=3}^{\infty} \frac{1}{n^2}$$ which still converges.

An important point in this last example is that the convergence of a series doesn't depend on the first term, or the first two terms, or even the first thousand terms. So, for example, the comparison test conclusions still hold if you only know $0 \leq a_n \leq b_n$ for all n bigger than some starting point ($n=3$ above). In other words, we need $0 \leq a_n \leq b_n$ eventually.

HELPFUL NOTE: From chapter 8, we learned that the following inequalities all hold eventually.

(any constant) $\leq \ln(n) \leq n^{(\text{positive power})} \leq (\text{base} > 1)^n$ ... and further... $\leq n! \leq n^n$.

2 The details involved in the comparison test seem too messy. Ideally, we shouldn't need to figure out the extra constants involved and just focus on the comparison. The limit comparison test allows for this:

LIMIT COMPARISON TEST: Suppose $a_n \geq 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same behavior (both converge or both diverge).

This again does a comparison, and says that $a_n \approx b_n$ (or $a_n \approx L \cdot b_n$) so $\sum_{n=1}^{\infty} a_n \approx L\sum_{n=1}^{\infty} b_n$, at least for n large.

EXAMPLES: Determine convergence

① $\sum_{n=0}^{\infty} \frac{1}{n^2-3}$

We expected this to behave like $\frac{1}{n^2}$, so we do this comparison: $L = \lim_{n\to\infty} \frac{\frac{1}{n^2-3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-3} = 1$. So, by limit comparison, this series converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also does (p-series, $p=2$).

② $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n^2+3n+5}}$

Here the denominator is roughly $\sqrt{n^2} = n$ and the numerator is constant, so we compare to $\frac{1}{n}$. Then

$L = \lim_{n\to\infty} \frac{\frac{4}{\sqrt{n^2+3n+5}}}{1/n} = \lim_{n\to\infty} \frac{4n}{\sqrt{n^2+3n+5}} = \lim_{n\to\infty} \frac{4}{\sqrt{1+3/n+5/n^2}} = \frac{4}{\sqrt{1+0+0}} = 4$. This is neither 0 nor ∞, so by limit comparison this series diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ does, too.

③ $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3}$

We know $0 \leq \sin^2(n) \leq 1$, so we expect the series to behave like $\frac{1}{n^3}$. Applying limit comparison gives

$L = \lim_{n\to\infty} \frac{\sin^2(n)/n^3}{1/n^3} = \lim_{n\to\infty} \sin^2(n)$ DNE. So, actually, limit comparison doesn't help here. Instead, we should use ordinary comparison: $0 \leq \frac{\sin^2(n)}{n^3} \leq \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3}$ does, too.

3 The issue with both comparison tests is that you need to create a comparison series. It would be nice to have a test that investigates the series in and of itself. This is what the ratio test achieves.

RATIO TEST: Suppose $a_n > 0$ and $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

i) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

ii) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

This test is saying that for n large, $\frac{a_{n+1}}{a_n} \approx r$, i.e. $a_{n+1} \approx r a_n$ which is saying that a_n is (roughly) geometric. As such, it is particularly useful when a_n involves n in an exponent or $n!$ or other factorial.

EXAMPLES: Determine convergence/divergence.

① $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

There's nothing obvious to compare to since there aren't any "ignorable" terms. If we try the ratio test:

$$r = \lim_{n\to\infty} \frac{\left(\frac{n+1}{2^{n+1}}\right)}{\left(\frac{n}{2^n}\right)} = \lim_{n\to\infty} \frac{(n+1)\cdot 2^n}{n \cdot 2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$ This is less than 1 so it converges.

② $\sum_{n=0}^{\infty} \frac{1}{n!}$

As suggested, use ratio test for factorials:

$$r = \lim_{n\to\infty} \frac{(1/(n+1)!)}{(1/n!)} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \cdots \cdot (n-2)(n-1)(n)}{1\cdot 2\cdot 3\cdot \cdots \cdot (n-1)(n)(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

So, by the ratio test, this series also converges.

③ $\sum_{n=1}^{\infty} \frac{3^n}{n^{10}}$

By the ratio test, $r = \lim_{n\to\infty} \frac{(3^{n+1}/(n+1)^{10})}{(3^n/n^{10})} = \lim_{n\to\infty} \frac{3^{n+1}\cdot n^{10}}{3^n\cdot (n+1)^{10}} = \lim_{n\to\infty} \frac{3n^{10}}{(n+1)^{10}} = \frac{3}{1} = 3 > 1.$

So this series diverges.

④ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

By the ratio test, $r = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2+1}}{\frac{1}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} = 1.$ But the ratio test has no conclusion when $r=1$. So we need another test. Instead, we do limit comparison with $\frac{1}{n^2}$ to get

$L = \lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$, so they have the same behavior. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2$), so by limit comparison $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ does, too.