§9.5

Main Concepts

1. Absolute convergence.
2. Alternating series test.
3. Conditional convergence.

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1. All of the tests of the last two sections require $a_n \geq 0$. How do we handle series with negative terms?

Theorem: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Examples: Apply the above theorem to each series.

① $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$.

After taking absolute values, we have $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$. But $|\sin(n)| \leq 1$, so $0 \leq \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$, and by ordinary comparison, $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ converges. By the above theorem, $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges.

② $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Notice $(-1)^{n+1} = \pm 1$, so $\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n}$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So the above theorem doesn't apply.

③ $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}$

Taking absolute values: $\left|\frac{(-5)^n}{n!}\right| = \frac{5^n}{n!}$ which involves both powers and factorials, so we use the ratio test:

$r = \lim_{n\to\infty} \frac{5^{n+1}/(n+1)!}{5^n/n!} = \lim_{n\to\infty} \frac{5^{n+1} \cdot n!}{5^n \cdot (n+1)!} = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1$. So, by the ratio test, $\sum_{n=0}^{\infty} \frac{5^n}{n!}$ converges, and by the above theorem, $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}$ converges, too.

This last case is common enough that we wrap it in a single test:

Absolute Ratio Test: Suppose $a_n \neq 0$ and $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

i) If $r < 1$, then $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ both converge.

ii) If $r > 1$, then both of these sums diverge instead.

2. How, then, can we handle example ② above? That series is a particular example of an *alternating series*— any series that switches signs on *every* term. For these, we have an independent test, abbreviated AST:

Alternating Series Test: Consider the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots$. If all of the following holds, the series converges:

i) $a_n \geq 0$ for all n (it's really alternating)

ii) $a_{n+1} \leq a_n$ for all n (the terms are decreasing in magnitude)

iii) $\lim_{n\to\infty} a_n = 0$.

Examples: Determine convergence/divergence.

① $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

This is alternating; for $a_n = \frac{1}{n}$ this is $a_1 - a_2 + a_3 - a_4 + \cdots$. Clearly $a_n \geq 0$ and $a_{n+1} \leq a_n$ since $0 \leq \frac{1}{n+1} \leq \frac{1}{n}$ for all n. Also, $\lim_{n\to\infty} \frac{1}{n} = 0$ as desired. So the series converges

by AST.

② $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{\sqrt{n+1}}$

This is also alternating since $\cos(0)=1$, $\cos(\pi)=-1$, $\cos(2\pi)=1$, ... So for $a_n = \frac{1}{\sqrt{n+1}}$ this is $a_0 - a_1 + a_2 - a_3 + a_4 - \dots$. Again $0 \le \frac{1}{\sqrt{n+2}} \le \frac{1}{\sqrt{n+1}}$, and $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ so the series again converges by AST.

3. Finally, a note on terminology. We've found that series fall into 3 categories:

DEFINITION: If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Finally, of course, it is possible that $\sum_{n=1}^{\infty} a_n$ diverges, giving case 3.

EXAMPLES: Decide if each series converges absolutely, conditionally, or not at all.

① $\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$

This is already positive, and it converges by ordinary comparison with $\frac{2}{n^2}$, so it converges absolutely.

② $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

We saw that this converges by AST, but that $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So this series converges conditionally.

③ $\sum_{n=0}^{\infty} (-1)^n$.

This is an alternating series, but it's also geometric with $r=-1$, so it diverges.

In summary, we have the following process for determining if a series converges: Given $\sum_{n=0}^{\infty} a_n$,

(I) If $\lim_{n\to\infty} a_n \neq 0$, the series diverges (Test for Divergence).

(II) If $a_n \geq 0$ [eventually]:

i) If a_n involves $n!$ or n in an exponent, try the Ratio Test.

ii) Otherwise, try using a Comparison test with some p-series.

iii) If none of this worked, try the Integral Test.

(III) If a_n switches signs:

i) If a_n alternates, try AST.

ii) Otherwise, decide if $\sum_{n=0}^{\infty} |a_n|$ converges (start over the flowchart with $|a_n|$).

(IV) Failing all else, compute partial sums and look for patterns (Telescoping, etc.)