§9.6

MAIN CONCEPTS

1. Power series
2. Radius and interval of convergence.

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1. So far, our series have consisted of numbers. We'd like to extend this to allow an infinite series with variables. The simplest of these are power series, which involve nonnegative powers of x or a shift of x.

DEFINITION: A power series in x, centered at $x=a$, is a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$.

EXAMPLES:

① $\sum_{n=0}^{\infty} x^n$ is a power series, obtained by taking $a=0$ and $c_n = 1$ for all n. In fact, if $-1<x<1$, this series converges to $\frac{1}{1-x}$ since it's geometric.

② $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a power series with $a=0$ and $c_n = \frac{1}{n!}$ for each n.

③ $\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n}$ is a power series, though it's not obvious as written. Rewriting it gives:

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{[2(x+1/2)]^n}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n}(x+1/2)^n$$ so $a=-\frac{1}{2}$ and $c_n = \frac{2^n}{n}$ for $n \geq 1$ and $c_0 = 0$.

2. We can ask, for a power series, which values of x makes it converge. For example ① we solved this: $-1<x<1$. In general, we have:

THEOREM: The series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for certain values of x, called the interval of convergence. It must take one of the following forms:

(i) The interval of convergence is the single point $x=a$.
(ii) The interval of convergence is $(a-R, a+R)$, possibly including the endpoints.
(iii) The interval of convergence is $(-\infty, \infty)$.

The interval has radius of convergence (i) $R=0$, (ii) R, and (iii) $R=\infty$ in each case.

Most often, the (absolute) ratio test can be used to find R and the interval.

EXAMPLES: Find the interval/radius of convergence of each power series.

① $\sum_{n=0}^{\infty} x^n$

We already found the interval of convergence to be $(-1,1)$ with radius $R=1$.

② $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

By the ratio test: $r = \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty} \frac{|x|^{n+1}\cdot n!}{|x|^n \cdot (n+1)!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$. So this always converges, i.e. it has interval $(-\infty,\infty)$ and radius $R=\infty$.

③ $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$.

By the ratio test: $\lim_{n\to\infty} \left|\frac{(2x+1)^{n+1}/(n+1)}{(2x+1)^n/n}\right| = \lim_{n\to\infty} \frac{|2x+1|^{n+1}\cdot n}{|2x+1|^n\cdot(n+1)} = |2x+1| \lim_{n\to\infty} \frac{n}{n+1} = |2x+1|$. The ratio test tells us that the sum converges when $|2x+1|<1 \longrightarrow -1<2x+1<1 \longrightarrow -2<2x<0$

$\longrightarrow -1<x<0$.

When $|2x+1|>1$, the series diverges. However, when $|2x+1|=1$ the test is inconclusive. So we have to go back and test $x=-1$ and $x=0$ seperately.

$x=-1$: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-2+1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by AST.

$x=0$: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{(0+1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by p-series ($p=1$).

So the interval of convergence is $[-1, 0)$ with radius $R = 1/2$.

④ $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Same idea. Ratio test: $\lim_{n\to\infty} \frac{|x^{n+1}/(n+1)^2|}{|x^n/n^2|} = \lim_{n\to\infty} \frac{|x|^{n+1} \cdot n^2}{|x|^n \cdot (n+1)^2} = |x| \cdot \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |x| < 1$

$\rightarrow$ $-1 < x < 1$. Finally, test the endpoints:

$x=-1$: $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by AST

$x=1$: $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series ($p=2$).

So the interval of convergence is: $[-1, 1]$ with radius $R = 1$.

In summary: to find the interval of convergence, use the absolute ratio test, and go back to test the endpoints with the remaining tests.