§9.7

Main Concepts

1. Algebraic operations
2. Composition
3. Calculus operations
4. Key examples

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1. In this section, we describe many ways to transform power series. The first type of these are the algebraic ones — adding power series, multiplying them, etc. E.g. if $f(x)=\sum_{n=0}^{\infty} c_n x^n$ and $g(x)=\sum_{n=0}^{\infty} d_n x^n$, then

$$f(x)+g(x)=\sum_{n=0}^{\infty}(c_n+d_n)x^n \quad \text{and} \quad kf(x)=\sum_{n=0}^{\infty} kc_n x^n \text{ for a constant } k.$$

Multiplying is harder, but not impossible — you do infinite "FOIL"ing. I.e.

$$f(x)g(x)=(c_0+c_1x+c_2x^2+\cdots)(d_0+d_1x+d_2x^2+\cdots)=c_0(d_0+d_1x+d_2x^2)+c_1x(d_0+d_1x+\cdots)+\cdots$$
$$=(c_0d_0+c_0d_1x+\cdots)+(c_1d_0x+c_1d_1x^2+\cdots)+\cdots=c_0d_0+(c_0d_1+c_1d_0)x+\cdots$$

Example: Find a power series for $\frac{7}{1-x}$.

We know $\frac{1}{1-x}=1+x+x^2+x^3+\cdots$ for $-1<x<1$. Multiplying through by 7 gives $\frac{7}{1-x}=7+7x+7x^2+7x^3+\cdots$

2. To generate more complicated examples, we compose functions. Suppose $f(x)=\sum_{n=0}^{\infty} c_n x^n$, for $-R<x<R$. Then $f(g(x))=\sum_{n=0}^{\infty} c_n (g(x))^n$ for $-R<g(x)<R$.

Examples: Find power series for the stated functions.

① $\frac{1}{1-x^2}$.

Plug in x^2 for x in $\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n$, $-1<x<1$ to get:

$$\frac{1}{1-x^2}=1+(x^2)+(x^2)^2+(x^2)^3+\cdots=\sum_{n=0}^{\infty}(x^2)^n, \quad -1<x^2<1$$
$$\frac{1}{1-x^2}=1+x^2+x^4+x^6+x^8+\cdots=\sum_{n=0}^{\infty}x^{2n}, \quad -1<x<1$$

② $\frac{1}{1-3x}$.

Plug in $3x$: $\frac{1}{1-3x}=1+(3x)+(3x)^2+\cdots=1+3x+9x^2+27x^3+81x^4+\cdots$ for $-1<3x<1$

i.e. $-\frac{1}{3}<x<\frac{1}{3}$.

③ $\frac{1}{2-x}$.

Factor out $\frac{1}{2}$: $\frac{1}{2-x}=\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}$. Then plug in $\frac{x}{2}$ to get

$$\frac{1}{2-x}=\frac{1}{2}\left(1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\cdots\right)=\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}+\cdots \text{ for } -1<\frac{x}{2}<1 \rightarrow -2<x<2.$$

Alternatively, $\frac{1}{2-x}=\frac{1}{1-(x-1)}=1+(x-1)+(x-1)^2+(x-1)^3+\cdots$ for $-1<x-1<1 \rightarrow 0<x<2$.

Both of these are valid power series for $\frac{1}{2-x}$, just with different centers and radii.

3. We can also do calculus with power series. If $f(x)=\sum_{n=0}^{\infty} c_n x^n$ then $f'(x)=\sum_{n=0}^{\infty} n c_n x^{n-1}$ and

$$\int f(x)\,dx=C+\sum_{n=0}^{\infty}\frac{c_n}{n+1}x^{n+1},$$ exactly as you would differentiate/integrate normally.

Importantly, these operations do not change the radius of convergence.

EXAMPLES: Find power series for each function.

① $\frac{1}{(1-x)^2}$.

We could multiply $\frac{1}{1-x}$ with itself, but multiplying power series is hard. Instead, notice

$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+x^5+x^6+\cdots$ for $-1<x<1$ and differentiate:

$-(1-x)^{-2}\cdot(-1) = 1+2x+3x^2+4x^3+5x^4+\cdots$ for $-1<x<1$, i.e.

$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+5x^4+\cdots$ for $-1<x<1$.

② $\ln(1-x)$.

This time we integrate:

$\int \frac{1}{1-x}dx = \int (1+x+x^2+x^3+\cdots)dx$ for $-1<x<1$

$-\ln|1-x| = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ for $-1<x<1$. Since $x<1$, $1-x>0$, so $|1-x|=1-x$, i.e.

$-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ for $-1<x<1$. Plugging in $x=0$ gives

$-\ln(1-0) = C + 0 + \frac{0^2}{2} + \cdots \longrightarrow -\ln(1) = C \longrightarrow C = 0$. So, multiplying by -1 gives

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ for $-1<x<1$

[NOTE: this is also true for $x=-1$ by AST and continuity but I won't ask about this in this class].

④ We're building a repetoire of power series. We can do two more:

EXAMPLES: Find more power series.

① $\arctan(x)$.

Start with $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$ for $-1<x<1$. Plug in $-x^2$ for x:

$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8-\cdots$ for $-1<x^2<1 \longleftrightarrow -1<x<1$. Now integrate:

$\arctan(x) = \int \frac{1}{1+x^2}dx = \int (1-x^2+x^4-x^6+\cdots)dx = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C$. Plug in $x=0$ to get

$C = \arctan(0) = 0$. So $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

② e^x

We'll do this backwards. Consider $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$ Differentiate to get

$$f'(x) = 1 + \frac{2x}{2} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \cdots$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$= f(x)$$

So $f'(x)=f(x)$. We can solve this differential equation to get $f(x) = Ce^x$ for some C. But

$C = C\cdot e^0 = f(0) = 1+0+0+0+\cdots = 1$ so $C=1$. I.e. $e^x = f(x) = 1+x+\frac{x^2}{2!}+\cdots$

In summary:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad -1<x<1$
- $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ $\quad -1<x\leq 1$
- $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ $\quad -1\leq x\leq 1$
- $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. $\quad$ all x.