§9.8

MAIN CONCEPT

[1] Taylor series. [2] Approximating using Taylor series. [3] Recovering derivatives.

[1] So far, we've found power series by "playing around," but we'd like a deterministic algorithm to find it, if possible. Suppose $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$. We can find c_0 by plugging in $x=0$: $f(0) = c_0 + 0 + 0 + 0 + \cdots \longrightarrow c_0 = f(0)$. Differentiating shifts the exponents down: $f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$. Then again plug in $x=0$: $f'(0) = c_1 + 0 + \cdots \longrightarrow c_1 = f'(0)$.

Now repeat: $f''(x) = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots \longrightarrow f''(0) = 2c_2 \longrightarrow c_2 = \frac{f''(0)}{2}$.

$f'''(x) = 6c_3 + 24c_4x + 60c_5x^2 + \cdots \longrightarrow f'''(0) = 6c_3 \longrightarrow c_3 = \frac{f'''(0)}{6}$

$f^{(4)}(x) = 24c_4 + 120c_5x + \cdots \longrightarrow c_4 = \frac{f^{(4)}(0)}{24}$.

We recognize the pattern: $c_n = \frac{f^{(n)}(0)}{n!}$. In fact, centering it at $x=a$ works similarly:

DEFINITION: The Taylor series of $f(x)$ centered at $x=a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. When $a=0$, we also call this the Maclaurin series of $f(x)$.

EXAMPLES: Compute each Taylor series requested.

① $f(x) = e^x$ centered at $a=0$ (Maclaurin).

We want to compute $f^{(n)}(0)$ for each n. Let's make a table:

n	0	1	2	3	⋯	← "n" just counts up
$f^{(n)}(x)$	e^x	e^x	e^x	e^x	⋯	← the n^{th} derivative of $f(x)$. Since $\frac{d}{dx}e^x = e^x$, this is straightforward.
$f^{(n)}(0)$	$e^0 = 1$	1	1	1	⋯	← evaluate at $x=0$.

So, the Taylor series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. [This matches §9.7.]

② $f(x) = \cos(x)$ centered at $a=0$.

Ditto:

n	0	1	2	3	4	5	6	7	8	⋯
$f^{(n)}(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$+\sin(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$\sin(x)$	$\cos(x)$	⋯
$f^{(n)}(0)$	$\cos(0) = 1$	$-\sin(0) = 0$	$-\cos(0) = -1$	0	1	0	−1	0	1	⋯

So the Maclaurin series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{1}{0!}x^0 + \frac{0}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \frac{-1}{6!}x^6 + \cdots$

$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$.

③ $f(x) = \sin(x)$ centered at $a=0$.

We can follow the same process as above, but instead let's use the ideas of §9.7. Since we know the series for $\cos(x)$, we can differentiate to get:

$-\sin(x) = \frac{d}{dx}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) = -\frac{1}{2!}(2x^1) + \frac{1}{4!}(4x^3) - \frac{1}{6!}(6x^5) + \cdots$

$= -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{1}{7!}x^7 - \frac{1}{9!}x^9 + \cdots$

Multiplying by -1 gives the Maclaurin series for $\sin(x)$: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

④ $f(x) = e^x$ centered at $a=1$.

Again, we make the table. But we already know what it looks like: $f^{(n)}(x) = e^x$ for all n because the derivative of e^x is itself. So $f^{(n)}(1) = e^1 = e$. Thus, the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots$$

⑤ $f(x) = \sqrt{x}$ centered at $a=1$, (first four terms only).

Table:

n	0	1	2	3	...
$f^{(n)}(x)$	$\sqrt{x} = x^{1/2}$	$\frac{1}{2}x^{-1/2}$	$-\frac{1}{4}x^{-3/2}$	$\frac{3}{8}x^{-5/2}$	...
$f^{(n)}(1)$	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	...

So, the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{1}{0!}(x-1)^0 + \frac{1/2}{1!}(x-1)^1 + \frac{-1/4}{2!}(x-1)^2 + \cdots$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + \cdots$$

2 One advantage of Taylor series is that polynomials are easy to evaluate and integrate. So, we can use them to approximate values. We will return later to the question of accuracy (how good are the approximations?).

EXAMPLES: Use Taylor series to approximate the given values as suggested.

① $e^{0.1}$ using three terms of the Maclaurin series for e^x.

Above, we solved $e^x = 1 + x + \frac{x^2}{2} + \cdots$, so $e^{0.1} \approx 1 + (0.1) + \frac{(0.1)^2}{2} = 1.105$.

[On a calculator, $e^{0.1} = 1.1051709\ldots$; so this is pretty good!]

② $\sqrt{1.3}$ using three terms of the Taylor series for $\sqrt{x}$ centered at $a=1$.

Same idea: $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \cdots$, so $\sqrt{1.3} \approx 1 + \frac{1}{2}(1.3-1) - \frac{1}{8}(1.3-1)^2$

$= 1 + (0.15) - (0.01125) = 1.13875$.

③ $\int_0^1 e^{x^3} dx$ using two (nonzero) terms of the Maclaurin series for e^{x^3}.

We don't have the Maclaurin series for e^{x^3}. But we can get it in one of two ways. Since we only need two terms, we can just compute part of the table: ~~$f(x) = e^{x^3}$ $f'(x) = 3x^2 e^{x^3}$~~

For $f(x) = e^{x^3}$:

n	0	1	2	3	...
$f^{(n)}(x)$	e^{x^3}	$3x^2e^{x^3}$	$(6x + 9x^4)e^{x^3}$	$(6 + 54x^3 + 27x^6)e^{x^3}$	...
$f^{(n)}(0)$	$e^0 = 1$	0	0	6	...

So, $e^{x^3} = 1 + \frac{6}{3!}x^3 + \cdots = 1 + x^3 + \cdots$. We also could have substituted x^3 into the Maclaurin series for e^x. Indeed $e^x = 1 + x + \cdots$, so $e^{x^3} = 1 + x^3 + \cdots$.

In either case, we can now approximate: $\int_0^1 e^{x^3} dx \approx \int_0^1 (1 + x^3) dx = x + \frac{x^4}{4}\Big|_0^1 = \left(1 + \frac{1}{4}\right) - \left(0 + \frac{0}{4}\right)$

$= 5/4 = 1.25$.

④ $\int_0^1 \cos(x^2) dx$ using two nonzero terms of the Maclaurin series for $\cos(x^2)$.

Same idea. We know $\cos(x) = 1 - \frac{x^2}{2} + \cdots$, so $\cos(x^2) = 1 - \frac{(x^2)^2}{2} + \cdots = 1 - \frac{x^4}{2} + \cdots$

So, $\int_0^1 \cos(x^2) dx \approx \int_0^1 \left(1 - \frac{x^4}{2}\right) dx = x - \frac{x^5}{10}\Big|_0^1 = \left(1 - \frac{1}{10}\right) - \left(0 - \frac{0}{10}\right) = \frac{9}{10}$.

3 Finally, we sometimes go the other way: use a Taylor series to compute derivatives. The idea is as follows: suppose we know (somehow) that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for some coefficients c_n.

We already saw that $c_n = \frac{f^{(n)}(0)}{n!}$, so working backwards gives $f^{(n)}(0) = c_n \cdot n!$.

EXAMPLES: Compute the specified derivatives using Taylor series.

① $f^{(20)}(0)$ for $f(x) = xe^{2x}$.

We don't want to compute 20 derivatives. But we can find the desired Maclaurin series by manipulating known ones. First, $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$; so $e^{2x} = \sum_{n=0}^{\infty} \frac{1}{n!}(2x)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$.

Multiplying by x gives: $f(x) = xe^{2x} = x\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+1}$.

On the other hand, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. The coefficients must match, so let's look at the coefficient of x^{20}. In the first series, this is when $n=19$: $\frac{2^{19}}{19!} x^{19+1} = \frac{2^{19}}{19!} x^{20}$.

In the second series, this is when $n=20$: $\frac{f^{(20)}(0)}{20!} x^{20}$. So, we get:

$$\frac{2^{19}}{19!} = \frac{f^{(20)}(0)}{20!} \longrightarrow f^{(20)}(0) = \frac{20! \cdot 2^{19}}{19!} = \boxed{20 \cdot 2^{19}}$$

② $f^{(18)}(0)$ for $f(x) = \frac{1}{(1-x)^2}$.

We found the Maclaurin series for $f(x)$ in §9.7 (though we didn't call it that). As a reminder, we did: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\xrightarrow{\text{differentiate}}$ $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$. Again, this also equals $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, so we'll compare coefficients of x^{18}. In the first (at $n=19$), we have $19x^{19-1} = 19x^{18}$; and in the second (at $n=18$) we have $\frac{f^{(18)}(0)}{18!} x^{18}$. So

$$\frac{f^{(18)}(0)}{18!} = 19 \longrightarrow \boxed{f^{(18)}(0) = 18! \cdot 19 = 19!}$$