# §9.9

MAIN CONCEPTS

1. Integral Test estimation of error
2. Alternating series error
3. Taylor Series error

---

1. NOTE: This isn't section §9.9 in the textbook. Instead, these notes address topics missed/skipped in §9.3, §9.5, §9.8, and §9.9.

So far, we've either summed a series exactly (geometric/telescoping/etc.) or we've only been able to determine convergence. If a series converges, the terms get smaller and smaller, so we suspect that we can approximate the sum by adding enough terms. Recall that for a series $\sum_{n=1}^{\infty} a_n$, we define $S_N = \sum_{n=1}^{N} a_n$ and $S = \lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} a_n$. Then the error or remainder is $S - S_N = \sum_{n=N+1}^{\infty} a_n$ (sometimes called $R_N$).

In the case of the integral test, we can determine this using the same strategy. Indeed, if $a_n \geq 0$, ~~$a_n$ is decreasing, and~~ $a_n = f(x)$ for some integrable function $f(x)$, ~~then~~ and $f(x)$ is decreasing, then $\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x)\,dx$. This gives us an estimate:

INTEGRAL TEST APPROXIMATION: When the integral test applies, $0 \leq S - S_N \leq \int_N^{\infty} f(x)\,dx$.

EXAMPLE: Find the sum of the first three terms of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and give bounds on the overall sum.

The problem requests $S_3 = \sum_{n=1}^{3} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36} \approx 1.36$. Then we know $0 \leq S - S_N \leq \int_3^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-x^{-1}\Big|_3^b\right] = \lim_{b\to\infty}\left[-\frac{1}{b} + \frac{1}{3}\right] = \frac{1}{3}$. So,

$$S_N \leq S \leq S_N + \frac{1}{3}, \quad \Rightarrow \quad \boxed{\frac{49}{36} \leq S \leq \frac{61}{36}} \text{ or } \boxed{\sim 1.36 \leq S \leq \sim 1.69}$$

We often use this test the other way: deciding how much error is "okay" and then deciding how many terms we need.

EXAMPLES: Decide the smallest value of $N$ so that $S_N \approx S$ with an error of at most 0.001.

① $\sum_{n=1}^{\infty} \frac{1}{n^2}$

We know $S - S_N \leq \int_N^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\Big|_N^b\right] = \lim_{b\to\infty}\left[-\frac{1}{b} + \frac{1}{N}\right] = \frac{1}{N}$. So, we want

$$\frac{1}{N} \leq 0.001 \quad \rightarrow \quad N \geq \frac{1}{0.001} = 1000.$$ So we need to add 1000 terms.

② $\sum_{n=1}^{\infty} \frac{240}{n^5}$

Same idea: $S - S_N \le \int_N^{\infty} \frac{240}{x^5}\,dx = \lim_{b\to\infty}\left[\frac{240}{(-4)}x^{-4}\Big|_N^b\right] = \lim_{b\to\infty}\left[-\frac{10}{b^4} + \frac{10}{N^4}\right] = \frac{10}{N^4}$

and so we want $\frac{10}{N^4} \le 0.001 \longrightarrow N^4 \ge \frac{10}{0.001} = 10000 \longrightarrow N \ge 10$,

In this case, we only need 10 terms (this seems reasonable since the terms get smaller faster, so fewer of them are "important" for the first few digits.

③ $\sum_{n=1}^{\infty} \frac{1}{e^n}$

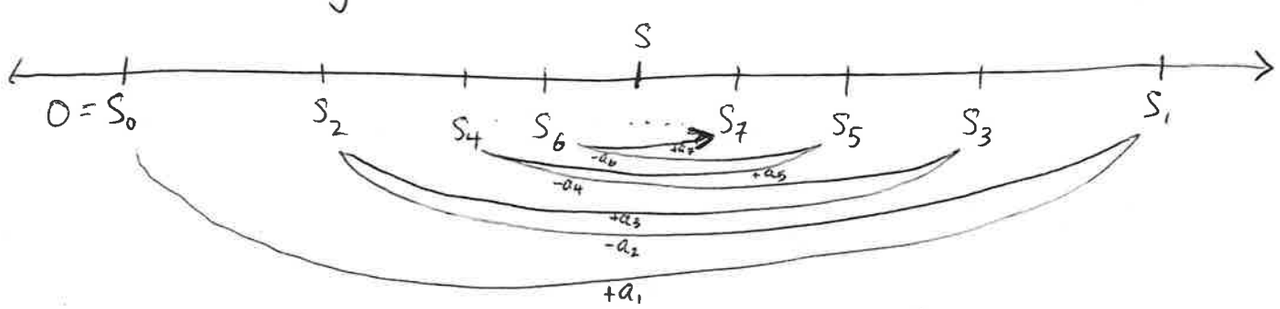
Ditto: $S - S_N \le \int_N^{\infty} \frac{1}{e^x}\,dx = \lim_{b\to\infty}\left(-e^{-x}\Big|_N^b\right) = \lim_{b\to\infty}\left(-\frac{1}{e^b} + \frac{1}{e^N}\right) = \frac{1}{e^N} \le 0.001$

$\longrightarrow e^N \ge 1000 \longrightarrow N \ge \ln(1000) \approx 6.91\ldots$ Since $N$ is an integer, we get $N \ge 7$, i.e, we need only the first 7 terms here.

Note that in ①, we need 1000 terms, and this demonstrates why computers can be useful in these estimations (it can sum 1000 terms incredibly quickly). Second, note that example ③ is actually geometric, so we know the sum is exactly $\frac{1/e}{1-1/e} = \frac{1}{e-1}$, so estimation is less useful here.

2. The limit comparison and ratio tests involve limits, and so the conclusions are only eventually true; as a result, we can't expect any error estimates. For convergence, we can ignore 10 or 10000 terms, but those terms could carry a lot of error.

Our next candidate, then, is the alternating series test. We gain some insight by using a number line and labelling partial sums. Consider $S_N = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_N$.




Our usual hypotheses are that $a_n$ is decreasing and $\lim_{n\to\infty} a_n = 0$. So, along the way, we jump forward by $a_1$, backwards by (the smaller) $a_2$, forwards by (the even smaller) $a_3$, etc. As a result, the partial sums always jump over the final sum $S$. We can therefore conclude that $|S - S_N| \le a_{N+1}$ for any $N$: $|S-S_N|$ is the distance from $S$ to $S_N$, and the next term jumps over $S$, so it has to be a bigger jump.

THEOREM [Alternating Series Estimation]: Suppose $a_n \ge 0$, $a_n$ is decreasing, and $\lim_{n\to\infty} a_n = 0$.

Let $S_N = \sum_{n=0}^{N} (-1)^n a_n$ and $S = \lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} (-1)^n a_n$. Then $|S - S_N| \le a_{N+1}$ for all $N$.

**EXAMPLE:** Give an upper bound on the error in $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ by adding 20 terms.

We have that $|S - S_{20}| \leq a_{21} = \frac{1}{21^2} \approx 0.00227$

As before, we often flip the question:

**EXAMPLES:** Decide the smallest value of $N$ so that $S_N \approx S$ with an error of at most 0.05.

① $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

We have $|S - S_N| \leq a_{N+1} = \frac{1}{(N+1)^2} \leq 0.05 \longrightarrow (N+1)^2 \geq \frac{1}{0.05} = 20$

$\longrightarrow N+1 \geq \sqrt{20} \longrightarrow N \geq \sqrt{20} - 1 \approx 3.47$. Since $N$ is a whole number, we need $\boxed{N \geq 4}$, i.e. 4 terms is enough.

② $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Same idea: $|S - S_N| \leq a_{N+1} = \frac{1}{(N+1)} \leq 0.05 \longrightarrow N+1 \geq \frac{1}{0.05} = 20 \longrightarrow \boxed{N \geq 19}$

③ $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

Ditto: $|S - S_N| \leq a_{N+1} = \frac{1}{\ln(N+1)} \leq 0.05 \longrightarrow \ln(N+1) \geq \frac{1}{0.05} = 20$

$\longrightarrow N+1 \geq e^{20} \longrightarrow N \geq e^{20} - 1 \approx 485,165,194.4098$

Again, $N$ is a whole number, so $\boxed{N \geq 485,165,195}$

This last example is demonstrating something important: conditionally convergent series may "barely converge" in the sense that millions of terms are needed for just the first digit. Even with a computer, directly summing this series isn't feasible.

3 Finally, we ask a related, but slightly distinct question: how well does a Taylor series approximate its function of definition? The main result, Taylor's theorem, is a result of the mean value theorem:

**THEOREM [Taylor]:** Let $f(x)$ be an $(n+1)$-times differentiable function, and let $T_N(x)$ be the degree $N$ Taylor polynomial centered at $x=a$, i.e. $T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$. Then, for each $x \neq a$, there is some $c$ between $x$ and $a$ such that $f(x) - T_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$.

There are a few ways to use this; first, we can actually prove that $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. What we've shown until now is that *if* $\sin(x)$ has a power series representation, then it must be the above, but it's possible that $\sin(x)$ has no power series at all.

But in this case, if $f(x) = \sin(x)$, then $f^{(n+1)}(x) = \pm\sin(x)$ or $\pm\cos(x)$ depending on $n$. So, the error $f(x) - T_N(x)$ is $\frac{f^{(N+1)}(c)}{(N+1)!}(x-0)^{N+1} = \frac{(\pm\sin(c) \text{ or } \pm\cos(c))}{(N+1)!} x^{N+1}$. So, the size of the error is $|f(x) - T_N(x)| = \frac{|\sin(c) \text{ or } \cos(c)|}{(N+1)!}|x|^{N+1} \leq \frac{|x|^{N+1}}{(N+1)!}$

since both $\sin(c)$ and $\cos(c)$ are in $[-1,1]$. But as $N$ approaches $\infty$, $\frac{x^{N+1}}{(N+1)!} \to 0$ by growth rate comparisons. I.e. the error goes to 0 and $f(x) = \lim_{N\to\infty} T_N(x)$ as claimed!

We can also use it as in [1] and [2] to estimate errors.

EXAMPLES: Estimate each quantity using 3 terms of an appropriate Taylor series, and bound the error.

① $\sqrt{e}$

This is $\sqrt{e} = e^{0.5}$, so let's use the Taylor series of $f(x) = e^x$. Then $T_2(x) = 1 + x + \frac{x^2}{2}$ since $f^{(n)}(x) = e^x$ for all $x$ and $f^{(n)}(0) = e^0 = 1$. So $e^{0.5} = f(0.5) \approx T_2(0.5) = 1.625$. Then,

$f(x) - T_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{e^c}{6}x^3$ for some $c$. At $x = 0.5$, we have:

$f(0.5) - T_2(0.5) = \frac{e^c}{6}(0.5)^3 \to \sqrt{e} - 1.625 = \frac{e^c}{48}$ for some $0 < c < 0.5$.

So $\frac{1}{48} = \frac{e^0}{48} < \sqrt{e} - 1.625 < \frac{e^{0.5}}{48} = \frac{\sqrt{e}}{48}$. The left estimate gives $1.625 + \frac{1}{48} < \sqrt{e}$, i.e. $1.645833\ldots < \sqrt{e}$. The right estimate gives $\sqrt{e} - 1.625 < \frac{\sqrt{e}}{48} \to \sqrt{e}\left(1 - \frac{1}{48}\right) < 1.625$

$\to \sqrt{e} < \frac{1.625}{(47/48)} = 1.659574\ldots$ I.e. $\sqrt{e} \in (1.645833\ldots, 1.659574\ldots)$

② $\ln(0.5)$.

Take $f(x) = \ln(x)$. This is simplest at $x = 1$ so we'll center it at $a = 1$. We'll compute some derivatives:

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f^{(n)}(x)$ | $\ln(x)$ | $\frac{1}{x}$ | $-\frac{1}{x^2}$ | $\frac{2}{x^3}$ |
| $f^{(n)}(1)$ | 0 | 1 | -1 | 2 |

So: $T_2(x) = (x-1) - \frac{(x-1)^2}{2}$ and $f(x) - T_2(x) = \frac{(2/c^3)}{3!}(x-1)^3$.

At $x = 0.5$ we get: $T_2(0.5) = (0.5-1) - \frac{(0.5-1)^2}{2} = 0.375$ and

$f(0.5) - T_2(0.5) = \frac{(2/c^3)}{6}(0.5-1)^3$ for some $0.5 < c < 1.5$

$\ln(1.5) - 0.375 = \frac{1}{24c^3}$.

We can be precise (as before) and plug in $c = 1$ and $c = 1.5$ to get the extreme cases. Since we're only trying to estimate the error, we can also say:

$\left|\ln(1.5) - 0.375\right| = \left|\frac{1}{24c^3}\right| = \frac{1}{24c^3} < \frac{1}{24(1)^3} = \frac{1}{24}$. So the error is at most $\frac{1}{24} \approx 0.042$.

Again, we can reverse the problem, though some creativity is needed sometimes.

EXAMPLES: Decide a value of $N$ that guarantees $|f(x) - T_N(x)| \le 0.001$ for the specified $f(x)$, $x$, and $a$.

① $f(x) = e^x$, $x = 1$, $a = 0$.

We have $|f(*) - T_N(1)| = \left|\frac{f^{(N+1)}(c)}{(N+1)!}(1-0)^{N+1}\right| = \frac{e^c}{(N+1)!}$ for some $0 < c < 1$,

So $e^c < e^1 = e$ and so it is enough if $\frac{e}{(N+1)!} \le 0.001 \to (N+1)! \ge \frac{e}{0.001} = 1000e$

I.e. $(N+1)! \geq 2718.28\ldots$ and the easiest way to solve this is plug in values of $N$. When $N=5$, $(N+1)! = 6! = 720$ isn't enough, but if $N=6$, $(N+1)! = 7! = 5040 \geq 2718.\ldots$ as required. So $\boxed{N \geq 6}$ terms is enough.

(2) $f(x) = \sqrt{x}$, $x = 0.9$, $a = 1$.

Same setup: $|f(x) - T_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!}(0.9-1)^{N+1} \right| = \frac{|f^{(N+1)}(c)|}{(N+1)!}(0.1)^{N+1}$ for some $0.9 < c < 1$.

We want this to be less than 0.001, so we'll begin by computing derivatives.

| $n$ | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|
| $f^{(n)}(x)$ | $\sqrt{x} = x^{1/2}$ | $\frac{1}{2}x^{-1/2}$ | $-\frac{1}{4}x^{-3/2}$ | $\frac{3}{8}x^{-5/2}$ | $-\frac{15}{16}x^{-7/2}$ | ... |

We want $|f^{(N+1)}(c)| \leq$ something so we ask what value of $c$ maximizes each. Most of these are negative powers of $x$, so they're maximized when $x=c$ is minimized. So we plug in $c = 0.9$ (the minimum possible) and compute $\frac{|f^{(N+1)}(0.9)|}{(N+1)!}(0.1)^{N+1}$.

| $N$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\frac{\lvert f^{(N+1)}(0.9)\rvert}{(N+1)!}(0.1)^{N+1}$ | $\approx 0.094868$ | $\approx 0.002635$ | $\approx 0.000049$ | $\approx 0.000002$ |

So $|f(x) - T_N(x)| = \frac{|f^{(N+1)}(c)|}{(N+1)!}(0.1)^{N+1} \leq \frac{|f^{(N+1)}(0.9)|}{(N+1)!}(0.1)^{N+1} \leq 0.000049 \leq 0.001$

(using $0.9 < c < 1$; for $N \geq 2$)

I.e. $\boxed{N \geq 2}$ suffices.

Notice that unlike (1), we never found $f^{(n)}(x)$ explicitly; we just computed the first few which was enough.

(3) $f(x) = \cos(x)$, $x = 1$, $a = 0$.

We know $f^{(n)}(x) = \pm\sin(x)$ or $\pm\cos(x)$ for all $x$ again, so $|f^{(N+1)}(c)| \leq 1$. Thus:

$$|f(1) - T_N(1)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!}(1-0)^{N+1} \right| = \frac{|f^{(N+1)}(c)|}{(N+1)!} \leq \frac{1}{(N+1)!} \leq 0.001$$

(want this)

so $(N+1)! \geq \frac{1}{0.001} = 1000$ and $N \geq 6$ suffices just as in (1).