Exercise (2.1.1).

Proof. Let us show that the constant sheaf $\mathscr{F}$ satisfies the universal property of being the sheafification of the constant presheaf $\mathscr{A}$. First, we exhibit the universal morphism θ. For $\emptyset \neq U \subseteq X$ open, define $\theta(U) : \mathscr{A}(U) = A \to \mathscr{F}(U)$ by:

$$\theta(U)(a)(x) = a$$

That is, we map an element a to the constant map $U \to A$ evaluating to a. Constant maps are always continuous, so this is a well-defined function, and $\theta(U)$ is clearly a homomorphism. Finally, θ is clearly compatible with restrictions, so it is a morphism of presheaves.

Now, suppose we have a sheaf $\mathscr{G}$ and a morphism $\varphi : \mathscr{A} \to \mathscr{G}$. We construct a factorization through θ. Fix an open subset $U \subseteq X$ and an element $f \in \mathscr{F}(U)$. For each $a \in A$, we get that $V_a = f^{-1}(a)$ is an open subset of U, since we've assumed f is continuous with A given the discrete topology. In fact, from this definition, it is clear that the collection $\{V_a\}$ is an open cover of U consisting of disjoint sets. Thus, if we define, for each a, the element $g_a = \varphi(V_a)(a) \in \mathscr{G}(V_a)$, then because $\mathscr{G}$ is a sheaf, we can glue these to a single element $g \in \mathscr{G}(U)$. Overall, let us define $\psi(U)(f) = g$.

Notice this is a well-defined function (there were, in the end, no choices made). Furthermore, each $\psi(U)$ is a homomorphism, for if $f_1, f_2 \in \mathscr{F}(U)$, then we can compute $\psi(U)(f_1+f_2)$ by gluing along the finer open cover given by $V_{a,b} = f_1^{-1}(a) \cap f_2^{-1}(b)$ for each $a, b \in A$. Finally, it is clear that ψ is compatible with restrictions.

Finally, it remains to show that $\varphi = \psi \circ \theta$. But this is clear, for if $a \in A$, U is open, and f_a denotes the constant function $U \to A$ with $f_a(x) = a$, then in the above notation $V_a = U$ and $V_b = \emptyset$ for $b \neq a$, so $g = g_a = \varphi(U)(a)$. Thus,

$$\psi(U)(\theta(U)(a)) = \psi(f_a) = g = \varphi(U)(a)$$

as desired. So, indeed we get that $\mathscr{F}$ satisfies the universal property, and so $\mathscr{F}$ is the sheafification of $\mathscr{A}$. □

Exercise (2.1.2).

Proof. Representing stalks at P by pairs $\langle U, s\rangle$ with U an open neighborhood of P and s a section on X up to equivalence under further restriction to neighborhoods of P, we have that

$$\begin{aligned}
\langle U, s\rangle \in \ker(\varphi_P) &\iff \varphi_P \langle U, s\rangle = 0 \\
&\iff \langle U, \varphi(U)(s)\rangle = \langle U, 0\rangle \\
&\iff (\exists V \subseteq U) : \varphi(U)(s)|_V = 0 \\
&\iff (\exists V \subseteq U) : \varphi(V)(s|_V) = 0 \\
&\iff (\exists V \subseteq U) : s|_V \in \ker(\varphi(V)) \\
&\iff \langle U, s\rangle \in (\ker\varphi)_P
\end{aligned}$$

Since stalks of a presheaf and stalks of its sheafification agree, this same computation works for the image (we use this fact in the reverse direction of the final biconditional).

Now, φ is injective iff $\ker\varphi = 0$, iff $(\ker\varphi)_P = 0$ for all P, iff $\ker(\varphi_P) = 0$ for all P, iff each φ_P is injective. For surjectivity, note by proposition 1.1 that the induced map $\operatorname{im}\varphi \to \mathscr{G}$ is an isomorphism iff it is an isomorphism on stalks. So, φ surjects iff $\operatorname{im}\varphi = \mathscr{G}$, iff $(\operatorname{im}\varphi)_P = \mathscr{G}_P$ for all P, iff $\operatorname{im}(\varphi_P) = \mathscr{G}_P$ for all P, iff φ_P is surjective for each P.

□

Exercise (2.1.3).

Proof. □

Exercise (2.1.4).

Proof. □

Exercise (2.1.5).

Proof. □

Exercise (2.1.6).

Proof. □

Exercise (2.1.7).

Proof. □

Exercise (2.1.8).

Proof. □

Exercise (2.1.9).

Proof. □