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The problem

Exercise. *Let $R = \prod_p \mathbb{F}_p$, with p ranging over the set of all prime numbers. Prove that R has a maximal ideal $\mathfrak{m}$ for which the field $R/\mathfrak{m}$ has characteristic zero and contains an algebraic closure of $\mathbb{Q}$.*

Before attacking it directly, let's develop some theory.

Filters

Throughout, let X be a set and $\mathcal{P}(X)$ denote its powerset.

Definition. *We say a set $\mathcal{F} \subseteq \mathcal{P}(X)$ is a filter on X if*

(a) $X \in \mathcal{F}$,

(b) If $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$, and

(c) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

We say that $\mathcal{F}$ is proper if $\mathcal{F} \neq \mathcal{P}(X)$. We say that $\mathcal{F}$ is an ultrafilter if it is maximal among proper filters.

Lemma. *$\mathcal{P}(X)$ is a filter on X.*

Proof. Obvious. □

Lemma. *Let $\{F_i\}_i$ be a collection of filters on X, indexed by i in some index set. Then $\bigcap_i F_i$ is a filter.*

Proof. Let $F = \bigcap_i F_i$. Since $X \in F_i$ for all i, $X \in F$. If $A \subseteq B$ and $A \in F$, then $A \in F_i$ for all i, so $B \in F_i$ for all i, whence $B \in F$. Finally, if $A, B \in F$, then $A, B \in F_i$ for all i, so $A \cap B \in F_i$ for all i, whence $A \cap B \in F$. □

Definition. *For $S \subseteq \mathcal{P}(X)$, let $\bar{S}$ denote the intersection of all filters containing S. We'll call this the filter generated by S. The previous two lemmas imply that this is indeed a filter.*

Definition. *Let $S \subseteq \mathcal{P}(X)$. We'll call $\hat{S} = \{A \subseteq X \mid \exists B \in S : A \subseteq B\}$ the upward closure of S. We'll call $S_\cap = \{A_1 \cap \cdots \cap A_n \mid A_1, \ldots, A_n \in S\}$ the finite intersection closure of S.*

Lemma. *Let $\emptyset \neq S \subseteq \mathcal{P}(X)$. Then $\bar{S} = (\hat{S})_\cap$.*

Proof. ($\subseteq$). For this containment, it suffices to show that $(\hat{S})_\cap$ is a filter containing S. Clearly it contains S, so we need to show it is a filter. It is clear that $X \in (\hat{S})_\cap$ since $X \in \hat{S}$ since S is nonempty.

Suppose now that $A \in (\hat{S})_\cap$ and that $B \supseteq A$. Then $A = A_1 \cap \cdots \cap A_n$ for some $A_1, \ldots, A_n \in \hat{S}$ and $A_i \supseteq C_i$ for some $C_1, \ldots, C_n \in S$. Then $B \cup C_i \in \hat{S}$ for each i since $B \cup C_i \supseteq C_i$, and

$$(B \cup C_1) \cap \cdots \cap (B \cup C_n) = B \cup (C_1 \cap \cdots \cap C_n) = B$$

since $C_1 \cap \cdots \cap C_n \subseteq A_1 \cap \cdots \cap A_n = A \subseteq B$. This shows that $B \in (\hat{S})_\cap$.

Second, if $A_1 \cap \cdots \cap A_n, A_{n+1} \cap \cdots \cap A_{n+m} \in (\hat{S})_\cap$, then it is immediate that their intersection is $A_1 \cap \cdots \cap A_{n+m} \in (\hat{S})_\cap$. This completes the argument that $(\hat{S})_\cap$ is a filter.

($\supseteq$). Let $A_1 \cap \cdots \cap A_n \in (\hat{S})_\cap$. As before, $A_i \supseteq B_i$ for some $B_1, \ldots, B_n \in S$. Then, $\bar{S}$ is a filter containing S, so it contains each B_i, so it contains each A_i, and so it contains their intersection. □

Lemma. *Let $S \subseteq \mathcal{P}(X)$ such that $B_1 \cap \cdots \cap B_n \neq \emptyset$ for every n-tuple $B_1, \ldots, B_n \in S$. Then the filter generated by S is proper.*

Proof. It suffices to show that $\emptyset \notin \bar{S}$. So, it suffices to show that $A_1 \cap \cdots \cap A_n \neq \emptyset$ for each n-tuple $A_1, \ldots, A_n \in \hat{S}$. But each A_i contains a $B_i \in S$, so $A_1 \cap \cdots \cap A_n \supseteq B_1 \cap \cdots \cap B_n \neq \emptyset$, so we're done. $\square$

Lemma. *Every proper filter is contained in an ultrafilter.*

Proof. Let F be a proper filter, and consider the collection of proper filters containing F. This is clearly nonempty since it contains F itself, and is partially ordered by inclusion. Suppose $\{F_i\}$ is a nonempty chain, and let $F = \bigcup_i F_i$. Then I claim that F is a proper filter containing each F_i. Indeed, since the chain is nonempty, $X \in F$ since it's in each F_i (and there's at least one). If $A \subseteq B$ and $A \in F$, then $A \in F_i$ for some i, whence $B \in F_i$, so $B \in F$. If $A, B \in F$, then $A \in F_i$ for some i and $B \in F_j$ for some j. Since this is a chain, we have WLOG that $F_i \subseteq F_j$, so $A, B \in F_j$, whence $A \cap B \in F_j$, so $A \cap B \in F$ as desired.

So, by Zorn's lemma, this collection has a maximal element G. That is, G is maximal among filters containing F. In fact, this implies that G is maximal among all filters, for if $G' \supseteq G$, then $G' \supseteq F$, so G' is in the above collection, and the maximality of G implies $G = G'$. So, G is in fact an ultrafilter. $\square$

Combining the previous two lemmas tells us that if S has nonempty finite intersections, then it's contained in an ultrafilter.

Lemma. *If $A \subseteq X$ and F is an ultrafilter, then exactly one of $A, X \setminus A \in F$.*

Proof. First, it is clear that F cannot contain both $A, X \setminus A$. Indeed, if it did, F would contain the intersection, which is empty. But then F contains all of its supersets, so F is not proper.

If $A \in F$, then we're done, so suppose $A \notin F$. Let $S = F \cup \{A\}$. Then $\bar{S}$ is a filter properly containing F, but F is an ultrafilter, so $\bar{S} = \mathcal{P}(X)$. In particular, $\emptyset \in \bar{S} = (\hat{S})_\cap$. So, by the above lemma, there exist $B_1, \ldots, B_n \in S$ such that $B_1 \cap \cdots \cap B_n = \emptyset$. If all of these were in F, then F would contain the empty set and not be proper. So, at least one of these must be A itself, and after removing redundancies and recognizing that F is closed under finite intersections, we conclude that $A \cap B = \emptyset$ for some $B \in F$. Hence, $B \subseteq X \setminus A$, and since F is a filter, $X \setminus A \in F$ as claimed. $\square$

Returning to the problem

Let's return to the problem.

Theorem. *Let $R = \prod_p \mathbb{F}_p$, with p ranging over the set of all prime numbers. Then R has a maximal ideal $\mathfrak{m}$ for which the field $R/\mathfrak{m}$ has characteristic zero and contains an algebraic closure of $\mathbb{Q}$.*

Proof. Let $X = \operatorname{Spec} \mathbb{Z}$. For $f \in \mathbb{Z}[x]$ nonconstant and monic, let

$$Z_f = \{p \in X \mid f \text{ has a root in } \mathbb{F}_p\}$$

Then, take

$$S = \{Z_f \mid f \in \mathbb{Z}[x] \text{ nonconstant and monic.}\}$$

I claim first that for $A_1, \ldots, A_n \in S$, $A_1 \cap \cdots \cap A_n \neq \emptyset$. In other words, $Z_{f_1} \cap \cdots \cap Z_{f_n} \neq \emptyset$. We may assume that each f_i is irreducible, since a root of an irreducible factor of f_i is also a root of f_i. Since we're in characteristic zero, they're each separable. Now, choose f to be the least common multiple of $f_1, \ldots, f_n$. Then, f is also separable, so by Chebotarev, f splits completely in some $\mathbb{F}_p[x]$. I.e. f_i has a root modulo p for each i, so $p \in Z_{f_1} \cap \cdots \cap Z_{f_n}$.

So, by the corresponding lemma in the previous section, $\bar{S}$ is a proper filter, and so $\bar{S} \subseteq F$ for some ultrafilter F. An element $\alpha \in R$ is a sequence $(\alpha_p)_{p\in X}$ such that $\alpha_p \in \mathbb{F}_p$. Define

$$Y_\alpha = \{p \in X \mid \alpha_p = 0\}$$

and

$$\mathfrak{m} = \{\alpha \in R \mid Y_\alpha \in F\}$$

To finish, we should show that $\mathfrak{m}$ is a maximal ideal and that $R/\mathfrak{m}$ contains an algebraic closure of $\mathbb{Q}$.

First, suppose $\alpha, \beta \in \mathfrak{m}$. Then $Y_{\alpha+\beta} \supseteq Y_\alpha \cap Y_\beta$, since $\alpha_p = \beta_p = 0$ means $(\alpha + \beta)_p = 0$. But F is a filter, so $Y_\alpha, Y_\beta \in F$ means $Y_\alpha \cap Y_\beta \in F$ and so $Y_{\alpha+\beta} \in F$, whence $\alpha + \beta \in \mathfrak{m}$. Similarly, if $\alpha \in \mathfrak{m}$ and $\beta \in R$, then $Y_{\alpha\beta} \supseteq Y_\alpha$ since $\alpha_p = 0$ means $(\alpha\beta)_p = 0$. So, again, $Y_{\alpha\beta} \in F$, so $\alpha\beta \in \mathfrak{m}$. This shows that $\mathfrak{m}$ is an ideal.

Now, suppose $\alpha \notin \mathfrak{m}$. Then $Y_\alpha \notin F$, and since it's an ultrafilter, the complement $X \setminus Y_\alpha$ is in F. Define

$$\beta_p = \begin{cases} 1 & \text{if } p \in Y_\alpha \\ 0 & \text{otherwise} \end{cases}$$

Then $Y_\beta = X \setminus Y_\alpha$, so $\beta \in \mathfrak{m}$. Then, $\alpha_p + \beta_p = (\alpha + \beta)_p \neq 0$ for any p, since exactly one of the summands is zero and one is nonzero for each p. Thus, $\alpha + \beta$ is a unit, showing that $\mathfrak{m}$ is maximal.

Finally, we should show that $R/\mathfrak{m}$ contains an algebraic closure of $\mathbb{Q}$. It suffices to show that $R/\mathfrak{m}$ contains all algebraic integers. That is, we should show that if $f(x) \in \mathbb{Z}[x]$ is monic and irreducible, then $R/\mathfrak{m}$ contains a root of f. Now, for each $p \in Z_f$, f has a root in $\mathbb{F}_p$, say α_p. For the remaining $p \in X$, define α_p arbitrarily. This specifies an element $\alpha \in R$. Further, we have that $Y_{f(\alpha)} \supseteq Z_f$ since

$$f(\alpha)_p = f(\alpha_p) = 0$$

for $p \in Z_f$. Hence $Y_{f(\alpha)} \in F$, so $f(\alpha) \in \mathfrak{m}$. But this shows that $f(\alpha) = 0$ in $R/\mathfrak{m}$ as desired, completing the proof. $\square$