Exercise (2.1.1).

Proof. Notice that each $f \in A$ with nonzero constant term is a unit. Indeed, after multiplying by an element of k, we may assume that the constant term is 1. I.e. $f = 1 + gT$ for some $g \in A$. Then, since A is T-adically complete, we have that

$$\sum_{k=0}^{\infty} (-gT)^k$$

converges to some $h \in A$. But then $hf = h(1 + gT) = 1$ directly, so f is invertible.

It is also clear that $\{0A, TA\} \subseteq \operatorname{Spec}(A)$, and I claim this list is complete. Indeed, suppose $P \in \operatorname{Spec}(A)$ is nonzero. Then it contains some nonzero element which can be factorized as $T^r f$ for some f with nonzero constant term. But we've shown that f is a unit, so P contains T^r, and since it's prime, it contains T. But TA is maximal, so $P = TA$ as desired. ☐

Exercise (2.1.2).

Proof. Let $\mathfrak{m} \in \operatorname{Spec}(B)$ be a closed point, i.e. a maximal ideal of B. Then we can take the composition:

$$A \xrightarrow{\varphi} B \to B/\mathfrak{m}$$

Then $B/\mathfrak{m}$ is a finitely generated k-algebra and a field, so it's a finite field extension (Zariski Lemma). So, each element of A maps to an algebraic element over k. I.e. the image of A in $B/\mathfrak{m}$ is a finitely generated k-algebra that is algebraic over k. Thus, it must be a field, so $A/\varphi^{-1}(\mathfrak{m})$ is a field, whence $\varphi^{-1}(\mathfrak{m})$ is maximal. So, $(\operatorname{Spec}\varphi)(\mathfrak{m}) = \varphi^{-1}(\mathfrak{m})$ is a closed point. ☐

Exercise (2.1.3).

Proof. Since $A/\mathfrak{m}$ is a finite field extension of $\mathbb{R}$, it is either $\mathbb{R}$ or $\mathbb{C}$. So, it injects into $\mathbb{C}$, giving:

$$\mathbb{R}[X, Y] \to A \to A/\mathfrak{m} \hookrightarrow \mathbb{C}$$

Then the images of $1, x, x^2$ in $\mathbb{C}$ are linearly dependent over $\mathbb{R}$, so there is some $a, b, c \in \mathbb{R}$ with

$$a + bx + cx^2 \in \mathfrak{m}$$

If c is nonzero, dividing by it gives the desired quadratic in x contained in $\mathfrak{m}$. Otherwise, we must have b nonzero, so dividing by b and squaring gives the desired quadratic. The same argument applies to y. The same argument applied to $1, x, y$ gives the final claimed relation:

$$f = \alpha x + \beta y + \gamma \in \mathfrak{m}$$

and if $(\alpha, \beta) = (0, 0)$, then $\gamma \in \mathfrak{m}$ and $\gamma \neq 0$, which contradicts properness of $\mathfrak{m}$.

We'd like to show $\mathfrak{m} = fA$. WLOG, assume $\beta \neq 0$. It suffices to show that $(f, X^2 + Y^2 + 1)$ is maximal in $\mathbb{R}[X, Y]$, where we also use f to denote $\alpha X + \beta Y + \gamma$. We'll take the quotient iteratively. First, we claim that $\mathbb{R}[X, Y]/(f) \cong \mathbb{R}[t]$. Indeed, consider the map $\psi : \mathbb{R}[X, Y] \to \mathbb{R}[t]$ given by:

$$X \mapsto t \text{ and } Y \mapsto -\frac{\alpha t + \gamma}{\beta}$$

Clearly $f \in \ker\psi$ and ψ surjects. Further, if $g \in \ker\psi$, then by polynomial division (in $\mathbb{R}[X][Y]$), we can write:

$$g(X, Y) = f(X, Y)q(X, Y) + h(X)$$

since β is a unit. Thus $\psi(h) = 0$, but $\psi(h) = h(t)$ is zero in $\mathbb{R}[t]$ iff it is zero in $\mathbb{R}[X]$. I.e. $g = fq \in (f)$. So, $\mathbb{R}[X, Y]/(f) \cong \mathbb{R}[t]$, and under this isomorphism, $X^2 + Y^2 + 1$ maps to a quadratic $p(t)$ (it is actually quadratic since the leading coefficient is positive, hence nonzero).

So, finally, to show that $(f, X^2 + Y^2 + 1)$ is maximal, it suffices to show that p is irreducible, i.e. that it has no root. Suppose it has a root u. Then we have the composite:

$$v : \mathbb{R}[X, Y] \to \mathbb{R}[t] \to \mathbb{R}$$

where the final map is evaluation $t \mapsto u$. Under the composition, we have $X^2 + Y^2 + 1 \mapsto p(u) = 0$, but this would mean that $0 = v(X)^2 + v(Y)^2 + 1 \geq 1$, which is a contradiction. Hence p is irreducible and $(f, X^2 + Y^2 + 1)$ is maximal, so that

$\mathfrak{m} = fA$.

The above proof shows in general that given $\alpha, \beta, \gamma \in \mathbb{R}$ with $(\alpha, \beta) \neq (0,0)$, that $(\alpha x + \beta y + \gamma)$ is maximal in A. Multiplying the vector (α, β, γ) does not change this ideal, so we indeed get a map $\mathbb{P}^2_{\mathbb{R}} \setminus \{(0:0:1)\} \to \operatorname{Spec}(A)$. We've also shown that this map surjects on the set of maximal ideals. So, we only need to show it injects. In other words, suppose $\mathfrak{m} = (\alpha x + \beta y + \gamma) = (\alpha' x + \beta' y + \gamma')$ for some $(\alpha : \beta : \gamma) \neq (\alpha' : \beta' : \gamma'] \in \mathbb{P}^2_{\mathbb{R}} \setminus \{[0:0:1]\}$. We have two cases. First, suppose (α, β) and (α', β') are independent in $\mathbb{R}^2$. Then they form a basis, so we can find $a, b, c, d \in \mathbb{R}$ with:

$$\begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $\mathfrak{m}$ contains:

$$a(\alpha x + \beta y + \gamma) + c(\alpha' x + \beta' y + \gamma') = x + r$$

for a constant $r \in \mathbb{R}$. Similarly, $\mathfrak{m}$ contains:

$$b(\alpha x + \beta y + \gamma) + d(\alpha' x + \beta' y + \gamma') = y + s$$

for a constant $s \in \mathbb{R}$. But then, in $A/\mathfrak{m}$ we have $x \mapsto -r$ and $y \mapsto -s$, which gives:

$$0 = x^2 + y^2 + 1 \mapsto r^2 + s^2 + 1 \geq 1$$

which is a contradiction.

So we must have that (α, β) and (α', β') are linearly dependent. Since they are nonzero, there is a constant c with $(\alpha, \beta) = c(\alpha', \beta')$. We cannot have $\gamma = c\gamma'$, since the two points are distinct in $\mathbb{P}^2_{\mathbb{R}}$. But then $\mathfrak{m}$ contains:

$$(\alpha x + \beta y + \gamma) - c(\alpha' x + \beta' y + \gamma') = \gamma - c\gamma'$$

which is nonzero, contradicting the fact that $\mathfrak{m}$ is proper. So, we've exhausted all possibilities and must have that the map is injective. Thus, the association $(\alpha : \beta : \gamma) \mapsto (\alpha x + \beta y + \gamma)A$ is a bijection between $\mathbb{P}^2_{\mathbb{R}} \setminus \{(0:0:1)\}$ and the maximal ideals of A.

As claimed, the map $\mathbb{R}[X] \to \mathbb{R}[X,Y] \to A$ is injective. Indeed, any element of the ideal $(X^2 + Y^2 + 1)$ necessarily has degree at least 2 in Y, while any polynomial in X has degree zero in Y. The morphism is also finite since we have a surjection of $\mathbb{R}[X]$-modules $\mathbb{R}[X]^2 \to A$ mapping (u, v) to $u + yv$. Indeed, given $g \in A$, we can lift it to $g(X,Y) \in \mathbb{R}[X,Y]$, and by polynomial division we can write

$$g(X,Y) = (X^2 + Y^2 + 1)q(X,Y) + Yr(X) + s(X)$$

so that after mapping into A we get:

$$g = yr(x) + s(x)$$

which is in the image of the $\mathbb{R}[X]$-module map above.

So now, if $\mathfrak{p}$ is a non-maximal prime of A, then I claim $\mathfrak{p} \cap \mathbb{R}[X]$ is a non-maximal prime of $\mathbb{R}[X]$. Indeed, let $\mathfrak{m}$ be a maximal ideal (strictly) containing $\mathfrak{p}$. Then, from the previous problem, $\mathbb{R}[X] \cap \mathfrak{m}$ is maximal and contains $\mathbb{R}[X] \cap \mathfrak{p}$. But we cannot have $\mathbb{R}[X] \cap \mathfrak{p} = \mathbb{R}[X] \cap \mathfrak{m}$ since A is an integral extension of $\mathbb{R}[X]$, which therefore satisfies incomparability. So $\mathbb{R}[X] \cap \mathfrak{p}$ is strictly contained in $\mathbb{R}[X] \cap \mathfrak{m}$, and therefore cannot be maximal.

But $\mathbb{R}[X]$ is a PID, so any nonzero prime is maximal, and therefore $\mathbb{R}[X] \cap \mathfrak{p} = 0$. Now let $g \in \mathfrak{p}$. Then $g \in A$, which is integral over $\mathbb{R}[X]$, so there is a relation:

$$g^n + a_{n-1}g^{n-1} + \cdots + a_0 = 0$$

where each $a_i \in \mathbb{R}[X]$. We may further assume that $n \geq 1$ is as small as possible. Now notice that a_0 is a multiple of g, so $a_0 \in \mathbb{R}[X] \cap \mathfrak{p} = 0$, so $a_0 = 0$. If $n > 1$, this gives

$$g^{n-1} + a_{n-1}g^{n-2} + \cdots + a_1 = 0$$

contradicting minimality of n. So instead we get $n = 1$, whence $g = -a_0 = 0$. So $\mathfrak{p} = 0$, and there is at most one non-maximal prime of A. $\square$

Exercise (2.1.4).

First, a geometric argument:

Proof. Notice that $\mathfrak{p}$ is an irreducible component of $\operatorname{Spec} A$. Then, if $f \in A_{\mathfrak{p}}$, then f is the germ of a function defined locally on $\mathfrak{p}$, and if $f \in \mathfrak{p}A_{\mathfrak{p}}$, then it vanishes on all of $\mathfrak{p}$. Hence, f vanishes everywhere, so by the Nullstellensatz, some power of it is zero.

Now, since A is reduced, only the zero function vanishes on all of $\operatorname{Spec} A$. Hence, if f is a zerodivisor, then there is some nonzero g with $fg = 0$, and g being nonzero implies that g does not vanish everywhere. Pick an irreducible component Z that g does not vanish on. Then

$$Z = Z \cap \operatorname{Spec}(A) = Z \cap V(fg) = Z \cap (V(f) \cup V(g)) = (Z \cap V(f)) \cup (Z \cap V(g))$$

By irreducibility, we must have $V(f) \supseteq Z$, so f vanishes on all of Z. □

Now, the formal algebra:

Proof. Note that $\mathfrak{p}A_{\mathfrak{p}}$ is the unique prime of $A_{\mathfrak{p}}$ since $\mathfrak{p}$ is minimal. Hence, the nilradical of $A_{\mathfrak{p}}$, as the intersection of the primes of $A_{\mathfrak{p}}$, is precisely $\mathfrak{p}A_{\mathfrak{p}}$. Hence, if $f \in \mathfrak{p}$ is nonzero, then its image is in $\mathfrak{p}A_{\mathfrak{p}}$, so there is some n such that $\frac{f^n}{1} = 0$. I.e. there is some $u \notin \mathfrak{p}$ with $f^n u = 0$, so f is a zerodivisor.

Now, let A be reduced, and let f be a zerodivisor, so there is some nonzero g with $fg = 0$. Then notice that g is not contained in every minimal prime, since otherwise it would be contained in

$$\bigcap_{\mathfrak{p} \text{ minimal prime}} \mathfrak{p} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \sqrt{0} = 0$$

since A is reduced. So, there is some minimal prime $\mathfrak{p}$ with $g \notin \mathfrak{p}$. But $fg = 0 \in \mathfrak{p}$, so $f \in \mathfrak{p}$ by primality. □

Exercise (2.1.5).

Proof. Note that injectivity at the beginning and surjectivity at the end are obvious. So, we need to see exactness in the middle. Further, any multiple of $P_1(T_1)$ is clearly in the kernel, so the image is contained in the kernel. Finally, let $f \in k[T_1, \ldots, T_n]$ be in the kernel. Write it as a polynomial in $T_2, \ldots, T_n$ with coefficients in $k[T_1]$, as:

$$f = \sum_i a(T_1) T^i$$

where i runs over multi-indices and T^i denotes $T_2^{i_2} \cdots T_n^{i_n}$. Then, the image in $k_1[T_2, \ldots, T_n]$ is:

$$\sum_i \overline{a(T_1)} T^i$$

which must be zero. I.e. each coefficient must be zero, so $a(T_1) \in P_1(T_1)k[T_1]$. Thus, factoring out $P_1(T_1)$ from each summand shows that f is a multiple of $P_1(T_1)$.

Now, we induct on n. Indeed, $P_1(T_1) \subseteq \mathfrak{m}$, so $\mathfrak{m}$ corresponds to a maximal ideal $\mathfrak{m}_1$ of $k_1[T_2, \ldots, T_n]$. Let $p_2(T_2)$ generate $\mathfrak{m}_1 \cap k_1[T_2]$. Then, lift this back to a polynomial $P_2(T_1, T_2) \in k[T_1, T_2]$. Notice that if $f \in k[T_1, T_2] \cap \mathfrak{m}$, then the image is in $\mathfrak{m}_1 \cap k_1[T_2]$, so f is contained in (P_1, P_2). Let $k_2 = k_1[T_2]/(p_2)$ and continue in this way to get the desired sequence. □

Exercise (2.1.6).

Proof. Let Z be a closed subspace of a quasi-compact space X. Let $\{U_i\}$ be an open cover of Z. Then, each U_i is of the form $U_i' \cap Z$ for some U_i' open in X. Notice that $\{U_i'\} \cup \{X \setminus Z\}$ is an open cover of X. This has a finite subcover, say $U_1', \ldots, U_n', X \setminus Z$ (if this last set is not needed, it doesn't hurt to have it anyway). Then $U_1, \ldots, U_n$ is a cover of Z, so indeed any open cover of Z has a finite subcover. I.e. Z is quasi-compact.

Now, let $\{U_i\}$ be an open cover of $\operatorname{Spec} A$. By definition, each U_i is the complement of $V(J_i)$ for some ideal J_i of A. Then, we get:

$$V\left(\sum_i J_i\right) = \bigcap_i V(J_i) = \operatorname{Spec}(A) \setminus \left(\bigcup_i U_i\right) = \emptyset$$

I.e. this ideal is not contained in any prime ideal, and so cannot be proper. I.e.

$$\sum_i J_i = A$$

whence we have an equation:

$$\sum_i j_i = 1$$

where $j_i \in J_i$ and $j_i = 0$ for all but finitely many indices. But then if S is the support of this sum (the set of i where $j_i \neq 0$), then:

$$\sum_{i\in S} j_i = 1 \implies \sum_{i \in S} J_i = A$$

So:

$$\bigcup_{i\in S} U_i = \operatorname{Spec}(A) \setminus \bigcap_{i \in S} V(J_i) = \operatorname{Spec}(A) \setminus V\left(\sum_{i\in S} J_i\right) = \operatorname{Spec}(A) \setminus V(A) = \operatorname{Spec}(A)$$

so that we have found a finite subcover. I.e. $\operatorname{Spec} A$ is quasi-compact. □

Exercise (2.1.7).

Proof. It suffices to describe the irreducible closed subsets of $\operatorname{Spec} \mathbb{Z}[T]$, which are of the form $V(P)$ for $P \in \operatorname{Spec} \mathbb{Z}[T]$. Indeed, any closed set is then a finite union of these irreducible closed sets. We will organize this by $\operatorname{ht} P$:

The only height zero prime is (0), and $V(0) = \mathbb{A}^1_{\mathbb{Z}}$ is the whole space.

The height one primes are of the form (f) where $f \in \mathbb{Z}[T]$ is irreducible (over $\mathbb{Q}$, say), or of the form (p) for p a rational prime. I'm not sure what to say (geometrically) about $V(f)$, but note that $V(p) = \mathbb{A}^1_{\mathbb{F}_p}$.

Finally, the height two primes are of the form (f, p) for p a prime and f irreducible when reduced mod p (i.e. over $\mathbb{F}_p$). These are the closed points $V(f, p)$. □

Exercise (2.1.8).

Proof. In other words, we would like to show that if $\varphi : A \to B$ is integral and $\mathfrak{m} \in \operatorname{Spec} B$ is maximal, then $\varphi^{-1}\mathfrak{m}$ is maximal. Consider the induced map $\varphi A/\varphi^{-1}\mathfrak{m} \to B/\mathfrak{m}$. This is an injective integral extension (since the reduction of the same polynomials work) from a domain to a field. So, we must have that $A/\varphi^{-1}\mathfrak{m}$ is also a field, i.e. $\varphi^{-1}\mathfrak{m}$ is maximal as claimed.

The second claim is false. Indeed, if $\mathscr{O}$ is the ring of integers in a number field K and p is an unramified non-inert prime of the extension $\mathscr{O}/\mathbb{Z}$ lying over p, then the preimage $\operatorname{Spec}(\varphi)^{-1}(p\mathbb{Z})$ is the set of primes lying over p, which is not a singleton by choice. I.e. the preimage of a closed point is not a closed point (while $\mathbb{Z} \to \mathscr{O}$ is integral).

Directly, each element $b \otimes 1$ is integral over $A_{\mathfrak{p}}$ since it is integral over A and the same polynomial works. So, the integral closure of the image contains each simple tensor $b \otimes 1$ and each $1 \otimes (a/s)$, so it must be the full ring.

That T is multiplicative is obvious. To see that $T^{-1}B \neq 0$, it suffices to notice that $0 \notin T$ since φ is injective and $0 \notin A \setminus \mathfrak{p}$. So, lastly we show the isomorphism. Notice that we have an A-bilinear map $B \times A_{\mathfrak{p}} \to T^{-1}B$ given by:

$$\left(b, \frac{a}{s}\right) \mapsto \frac{\varphi(a)b}{\varphi(s)}$$

which thus factors through a map $f : B \otimes_A A_{\mathfrak{p}} \to T^{-1}B$. We also have a map $B \to B \otimes_A A_{\mathfrak{p}}$ given by $b \mapsto b \otimes 1$. If $t = \varphi(a) \in T$, where $a \notin \mathfrak{p}$, then it maps to $t \otimes 1$, which satisfies:

$$(t \otimes 1)\left(1 \otimes \frac{1}{a}\right) = \left(\varphi(a) \otimes \frac{1}{a}\right) = a \cdot \left(1 \otimes \frac{1}{a}\right) = 1 \cdot 1$$

so that each element of T maps to something invertible. So, the map factors through a map $g : T^{-1}B \to B \otimes_A A_{\mathfrak{p}}$. These are the desired isomorphisms. Indeed, for $b/t \in T^{-1}B$:

$$f(g(b/t)) = f((b \otimes 1)(t \otimes 1)^{-1}) = \frac{b}{1}\left(\frac{t}{1}\right)^{-1} = b/t$$

so this composition is the identity. For a simple tensor $b \otimes (a/s)$, we have:

$$g(f(b \otimes (a/s))) = g\left(\frac{\varphi(a)b}{\varphi(s)}\right) = (\varphi(a)b \otimes 1)(\varphi(s) \otimes 1)^{-1} = a(b \otimes 1)[s(1 \otimes 1)]^{-1} = a(b \otimes 1)(1 \otimes (1/s)) = b \otimes (a/s)$$

and so is identity on simple tensors, and hence on all of the domain. So, f and g do indeed establish the isomorphism.

This gives the final claim. Indeed, given $\mathfrak{p} \in \operatorname{Spec} A$, we have the square:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \longrightarrow & B \otimes_A A_{\mathfrak{p}} = T^{-1}B \end{array}$$

Let $\mathfrak{m}$ be any closed point in $\operatorname{Spec} T^{-1}B$. Then, the image along the bottom arrow is a closed point in $\operatorname{Spec} A_{\mathfrak{p}}$, which therefore must be $\mathfrak{p}A_{\mathfrak{p}}$, as this is the unique closed point. Taking the further image along the localization map thus gives the ideal $\mathfrak{p} \in \operatorname{Spec} A$. But since this diagram commutes, the image in $\operatorname{Spec} B$ of $\mathfrak{m}$ gives a point of $\mathfrak{m}' \in \operatorname{Spec} B$ such that $\operatorname{Spec}(\varphi)(\mathfrak{m}') = \mathfrak{p}$ as desired. □

Exercise (2.1.9).

Proof. First, note that since A is a finite k-module, the extension is integral. Let $\mathfrak{p} \in \operatorname{Spec} A$; then it is contained in some maximal ideal $\mathfrak{m} \in \operatorname{Spec} A$. Then $\mathfrak{p} \cap k$ and $\mathfrak{m} \cap k$ must be primes in k, i.e. they must both be zero. So, they lie over the same prime, and since integral extensions satisfy incomparability, the containment $\mathfrak{p} \subseteq \mathfrak{m}$ implies $\mathfrak{p} = \mathfrak{m}$. So, indeed every prime is maximal.

Now, suppose $\operatorname{Spec} A$ is has at least $d + 1$ elements for $d = \dim_k(A)$. Then choose distinct prime (maximal) ideals $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots, \mathfrak{m}_{d+1}$. Let

$$I_r = \bigcap_{j=1}^{r} \mathfrak{m}_j$$

Then we have the descending chain:

$$A \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots I_{d+1}$$

Each of these containments must be proper. Indeed, otherwise we would have

$$\mathfrak{m}_1 \cdots \mathfrak{m}_r \subseteq I_r \subseteq \mathfrak{m}_{r+1}$$

But then, since $\mathfrak{m}_{r+1}$ is prime, it contains some $\mathfrak{m}_i$, whence $\mathfrak{m}_{r+1} = \mathfrak{m}_i$ by maximality, contrary to assumption.

On the other hand, each of these is a k-submodule of A, i.e. a k-vector space. Since the containments are proper, we must have that $\dim_k(I_{r+1}) < \dim_k(I_r)$. But $\dim_k(A) = d$, so this is a contradiction, as we cannot have a strictly decreasing sequence of length $d + 2$ among the integers $\{0, \ldots, d\}$. Thus, we must have $|\operatorname{Spec} A| \leq \dim_k(A)$.

Note that $k[T_1, \ldots, T_d]$ is a UFD, and a nonzero polynomial is a unit iff it is a constant. Hence, Euclid's proof shows that there are infinitely many prime elements and so infinitely many primes. Explicitly: note that T_1 is prime, so there is at least one nonzero prime. Suppose that $\pi_1, \ldots, \pi_n$ is a list of nonzero primes, so that none of them may be constant. Then $a = \pi_1 \cdots pi_n + 1$ factors as a nonempty product of primes since it has positive degree. None of these primes can be any π_i, else π_i divides 1. So, no finite list of primes of positive degree is complete, hence there are infinitely many primes.

We've already shown one direction, so now suppose that A is not a finite k-module. By Noether normalization, we can find a finite injective morphism $k[T_1, \ldots, T_d] \to A$. We must have $d \geq 1$ since we've assumed A is not finite over k. But now, the extension A over $k[T_1, \ldots, T_d]$ satisfies lying over, so we have at least one prime of A over each prime of $k[T_1, \ldots, T_d]$. In fact, this association is injective since each prime of A lies over a unique prime of $k[T_1, \ldots, T_d]$. So, $|\operatorname{Spec} A| \geq |\operatorname{Spec} k[T_1, \ldots, T_d]| = \infty$ as desired. □

Exercise (2.2.1).

Proof. Following the explicit construction given, the sheafification of the constant presheaf is the sheaf $\mathscr{A}^\dagger$ where $\mathscr{A}^\dagger(U)$ is the set of functions $f : U \to \bigsqcup_{x \in U} \mathscr{A}_x$ given locally by sections. Clearly $\mathscr{A}_x = A$, and the fact that f is locally given by sections means that for each $x \in U$, there is some open V and $a \in \mathscr{A}(V) = A$ with $f(x) = a_x = a$ for each $x \in V$. Another way to state this is that $f : U \to A$ is continuous when A is given the discrete topology. I.e. $\mathscr{A}^\dagger$ is the sheaf of continuous functions from subsets of X to A with the discrete topology, i.e. locally constant functions $X \to A$. Finally, we can interpret this as $\mathscr{A}^\dagger(U)$ being the direct product of copies of A in bijection with the connected components of U.

If every open subset of X is connected, then this description shows that $\mathscr{A} = \mathscr{A}^\dagger$, so $\mathscr{A}$ is a sheaf. Conversely, if there is some $U \subseteq X$ that is the disjoint union of open subsets V, W of X, then $\mathscr{A}$ is not a sheaf. Indeed, A is nontrivial, so we can choose two distinct elements $a, b \in A$. Then $a \in \mathscr{A}(V)$ and $b \in \mathscr{A}(W)$, and these sections agree on overlaps since $V \cap W = \emptyset$, but there is no $x \in \mathscr{A}(U)$ with $x|_V = x = a$ and $x|_W = x = b$. So, $\mathscr{A}$ does not satisfy gluing. □

Exercise (2.2.2).

Proof. Let $A = \{x \in X : s_x = t_x\}$. Suppose that $x \in A$. Then there is an open neighborhood U of x with $s|_U = t|_U$. But then for any $y \in U$, we have

$$s_y = (s|_U)_y = (t|_U)_y = t_y$$

so $y \in A$. Thus $U \subseteq A$ and we've shown that each $x \in A$ is contained in an open set contained in A, so A is open as claimed. □

Exercise (2.2.3).

Proof. I'll come back to this one. □

Exercise (2.2.4).

Proof. Name the maps:

$$0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$$

Recall that this implies that for each $x \in X$ that we have an exact sequence:

$$0 \to \mathscr{F}'_x \xrightarrow{\varphi_x} \mathscr{F}_x \xrightarrow{\psi_x} \mathscr{F}''_x \to 0$$

Now, exactness at $\mathscr{F}'(X)$ is obvious, since a map of sheaves is injective iff it is injective on each open set. So, we would like to show exactness at $\mathscr{F}(X)$. First, let $s \in \mathscr{F}'(X)$. Then for each $x \in X$ we have:

$$[\psi(X)(\varphi(X)(s))]_x = \psi_x([\varphi(X)(s)]_x) = \psi_x(\varphi_x(s_x)) = 0$$

since $\psi_x \circ \varphi_x = 0$. Hence the image of $\varphi(X)$ is contained in the kernel of $\psi(X)$. Conversely, suppose $s \in \ker(\psi(X)) \subseteq \mathscr{F}(X)$. Then, again, for each x, we have

$$\psi_x(s_x) = [\psi(X)(s)]_x = 0$$

So, $s_x \in \ker(\psi_x) = \operatorname{im}(\varphi_x)$. Hence, for each x we can find an open neighborhood $U(x)$ of x and a section $t(x) \in \mathscr{F}'(U(x))$ with $\varphi(U(x))(t(x)) = s|_{U(x)}$. We would like to glue these to a single global section, and so we need to see that they agree on overlaps.

To that end, let $x, y \in X$, let $a = t(x)|_{U(x)\cap U(y)}$, and let $b = t(y)|_{U(x)\cap U(y)}$. If $U(x) \cap U(y) = \emptyset$, then we must have $a = b$, and there is nothing to show. Otherwise, let $z \in U(x) \cap U(y)$. Then we have:

$$\varphi_z(a_z) = \varphi_z(t(x)_z) = [\varphi(U(x))(t(x))]_z = (s|_{U(x)})_z = s_z$$

and similarly:

$$\varphi_z(b_z) = \varphi_z(t(y)_z) = [\varphi(U(y))(t(y))]_z = (s|_{U(y)})_z = s_z$$

So, $\varphi_z(a_z) = \varphi_z(b_z)$ and by injectivity of φ_z we get $a_z = b_z$. Since this is true for all $z \in U(x) \cap U(y)$, we get that $a = b$, and so the $t(x)$ do indeed agree on overlaps. Hence they glue to a single section $t \in \mathscr{F}'(X)$ with $t|_{U(x)} = t(x)$.

But then we are finished, for if $x \in X$, then:

$$[\varphi(X)(t)]_x = \varphi_x(t_x) = \varphi_x((t|_{U(x)})_x) = \varphi_x(t(x)_x) = [\varphi(U(x))(t(x))]_x = (s|_{U(x)})_x = s_x$$

and so $\varphi(X)(t) = s$ since they are equal on all stalks. I.e. s is in the image of $\varphi(X)$. □

Exercise (2.2.5).

Proof. □

Exercise (2.2.6).

Proof. □

Exercise (2.2.7).

Proof. □

Exercise (2.2.8).

Proof. □

Exercise (2.2.9).

Proof. □

Exercise (2.2.10).

Proof. □

Exercise (2.2.11).

Proof. □

Exercise (2.2.12).

Proof. □

Exercise (2.2.13).

Proof. □

Exercise (2.2.14).

Proof. □

Exercise (2.3.1).

Proof. □

Exercise (2.3.2).

Proof. Let $i : U \to X$ be the inclusion map, so that we wish to show $i^\# : A \to B$ is flat. It suffices to show this locally. That is, for $\mathfrak{q} \in \operatorname{Spec} B = U$ and $\mathfrak{p} = i(\mathfrak{q}) \in \operatorname{Spec} A$, we wish to show that $A_\mathfrak{p} \to B_\mathfrak{q}$ is flat. But this is the map $i^\#_\mathfrak{q} : U_\mathfrak{q} \to X_{i(\mathfrak{q})}$, which is an isomorphism since U is an open subscheme. Isomorphisms are clearly flat, so the claim is shown. □

Exercise (2.3.3).

Proof. Notice that if $\mathfrak{p} \in F$, then $\mathfrak{p} \supseteq I$ tautologically, and so $\mathfrak{p} \in V(I)$ by definition. I.e. we immediately have $F \subseteq V(I)$, and since $V(I)$ is closed, this gives $\overline{F} \subseteq V(I)$. We wish to show the reverse.

Conversely, we have that $\overline{F}$ is a closed set, so it is of the form $V(J)$ for some ideal J. Then:

$$J \subseteq \sqrt{J} = \bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \overline{F}} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in F} \mathfrak{p} = I$$

where the inclusion comes from the fact that we are intersecting over a subset. So, by applying V, we get: $\overline{F} = V(J) \supseteq V(I)$, as desired.

Any prime $\mathfrak{p} \in \operatorname{im}(f)$ contains $\ker\varphi$, so $\operatorname{im}(f) \subseteq V(\ker\varphi)$, and since the latter is closed, we have $\overline{\operatorname{im}(f)} \subseteq V(\ker\varphi)$. Conversely, again write $\overline{\operatorname{im}(f)}$ as $V(J)$ for some $J \in \operatorname{Spec} A$. Then, for any prime $\mathfrak{p} \in \operatorname{Spec} B$, we have $f(\mathfrak{p}) \in V(J)$, i.e. $\varphi^{-1}\mathfrak{p} \supseteq J$. Thus, $\varphi(J) \subseteq \mathfrak{p}$. Since this is true of any prime, $\varphi(J)$ is contained in their intersection, i.e. $\varphi(J) \subseteq \sqrt{0}$, and so $J \subseteq \sqrt{\ker\varphi}$. But then $V(J) \supseteq V(\ker\varphi)$ as desired.

When we have $\varphi : A \to A_\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} A$, then on the one hand $\operatorname{im} f$ is precisely the set of primes contained in $\mathfrak{p}$. On the other hand, we have $\ker\varphi = \{a \in A : au = 0 \text{ for some } u \notin \mathfrak{p}\} \subseteq \mathfrak{p}$. So, there is some subideal I of $\mathfrak{p}$ such that the lattice of ideals containing I corresponds (bijectively, inclusion-preserving) to the closure of the lattice of ideals contained in $\mathfrak{p}$. □

Exercise (2.3.4).

Proof. □

Exercise (2.3.5).

Proof. Let $A = \mathscr{O}_Y(Y)$ be the global sections on Y and let $X = \operatorname{Spec} A$. We will show that $Y \cong X$, and hence is affine as claimed.

Notice that $\mathscr{O}_X(X) = A$. In particular, we have the identity ring homomorphism $i : \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)$, and this gives rise to a morphism $f : X \to Y$ of schemes with $\rho(f) = i$ since ρ is surjective by assumption. Similarly, since X is affine, we can apply Proposition 3.25 directly (with the roles of X and Y reversed). That is, for the identity map $j : \mathscr{O}_X(X) \to \mathscr{O}_Y(Y)$, we get a morphism $g : Y \to X$ with $\rho(g) = j$ since ρ is also surjective in this case. Now, we use functoriality of ρ. Namely, we have:

$$\rho(g \circ f) = \rho(f) \circ \rho(g) = i \circ j = \mathrm{id}_{\mathscr{O}_X(X)} = \rho(\mathrm{id}_X)$$

But since X is affine, ρ is also injective, so we get that $g \circ f = \mathrm{id}_X$. Similarly, we have

$$\rho(f \circ g) = \rho(g) \circ \rho(f) = j \circ i = \mathrm{id}_{\mathscr{O}_Y(Y)} = \rho(\mathrm{id}_Y)$$

However, we do not automatically know that ρ is injective in this case: the assumption only gives us this for maps $X \to Y$ with X affine, while we have a map $Y \to Y$ and are trying to show Y is affine. So, instead, let us work locally. Namely, let $U \subseteq Y$ be an affine chart, and let $\iota : U \hookrightarrow Y$ be the inclusion. Then,

$$\rho(f \circ g \circ \iota) = \rho(\iota) \circ \rho(g) \circ \rho(f) = \rho(\iota) \circ \mathrm{id}_{\mathscr{O}_Y(Y)} = \rho(\iota)$$

But now, we do have injectivity, since U is affine. I.e. we get $\iota = f \circ g \circ \iota$. Written alternatively, we have that $f \circ g$ and the identity map on Y agree when restricted to any affine chart U. Since these form a cover, we get that these global sections are equal everywhere, i.e. $f \circ g = \mathrm{id}_Y$. Thus, we have shown that f, g furnish an isomorphism between X and Y as desired. □

Exercise (2.3.6).

Proof. □

Exercise (2.3.7).

Proof. Note that $x \in X(k)$ means that $x : \operatorname{Spec} k \to X$ is a section of the structure morphism $X \to \operatorname{Spec} k$. Thus, $f(x)$ is shorthand for $f \circ x : \operatorname{Spec} k \to \mathbb{A}^n_k$. The identification of $\mathbb{A}^n_k(k)$ with k^n is as follows: given a section $t : \operatorname{Spec} k \to \mathbb{A}^n_k$, we get a map $t^\# : k[T_1, \ldots, T_n] \to k$, and we identify t with the tuple $(t^\#(T_1), \ldots, t^\#(T_n))$. So, in our case, $f(x)$ corresponds to the point whose ith coordinate is:

$$(f \circ x)^\#(T_i) = x^\#(f^\#(T_i)) = x^\#(\varphi(T_i)) = x^\#(f_i) = f_i(x)$$

as claimed. □

Exercise (2.3.8).

Proof. The first claim is immediate. Indeed, if $x \in X$, then since the $f_{i,x}$ generate $\mathscr{O}_{X,x}$, at least one of them cannot be in the maximal ideal of this local ring. That is, $f_i \notin \mathfrak{m}_{X,x}$ for some i. But then $x \in X_{f_i}$ for this i, so X is the union of the X_{f_i}.

Now, note that if $f \in \mathscr{O}_X(X)$ for any scheme X, then there is some $g \in \mathscr{O}_X(X_f)$ with $(f|_{X_f})g = 1$. That is, $f|_{X_f}$ is a unit. This is clear: for $p \in X_f$, we have f_p is invertible, so there is some open U containing p and some $g(p) \in \mathscr{O}_X(U)$ with $(f|_U)g(p) = 1$. Gluing these gives the g we seek.

For $i \in \{0, \ldots, n\}$, let $A_i = A[u_{ij}]$ for variables u_{ij} with $j \in \{0, \ldots, n\} \setminus \{i\}$. That is, each A_i is a polynomial ring over A in n variables. Define the ring homomorphisms $\varphi_i : A_i \to \mathscr{O}_{X_f}(X_f)$ by $\varphi_i(u_{ij}) = (f_i|_{X_{f_i}})^{-1}(f_j|_{X_{f_i}})$. By the note above, this is well-defined, since $f_i|_{X_{f_i}}$ is invertible. These induce morphisms $F_i : X_{f_i} \to \operatorname{Spec} A_i$.

Notice that

$$D_+(T_i) = \operatorname{Spec} A[T_0, \ldots, T_n]_{(T_i)} = \operatorname{Spec} A\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

which has a clear isomorphism with $\operatorname{Spec} A_i$ given by $u_{ij} \mapsto T_j/T_i$. Via these isomorphisms, we can extend our maps to $F_i : X_{f_i} \to \mathbb{P}^n_A$ by composing with the inclusion maps.

Finally, we will glue these maps to get a single morphism $f : X \to \mathbb{P}^n_A$. It is clear from the construction that f satisfies the desired properties.

In the case that A is a field and $x \in X(A)$ is a rational point, we can write $f_i(x)$ for the image of $f_{i,x}$ in $\mathscr{O}_{X,x}/\mathfrak{m}_{X,x} = A$. Then, via the identification of the A-rational points of $\mathbb{P}^n_A$ with equivalence classes of $(n+1)$-tuples in A, we have $f(x) = [f_0(x) : \cdots : f_n(x)]$. □

Exercise (2.3.9).

Proof. □

Exercise (2.3.10).

Proof. Let $f \in \mathscr{O}_X(X)$ be a global section, and let $U_i = D_+(T_i)$ so that $U_i \cong \operatorname{Spec} A[T_0/T_i, \dots, T_n/T_i]$. Fix some indices $i \neq j$. Then we have that the restriction of f to U_i can be expressed as a polynomial in the variables $T_0/T_i, \dots, T_n/T_j$, so after clearing denominators, we have:

$$g = T_i^a (f|_{U_i})$$

for some $g \in A[T_0, \dots, T_n]$ that is not a multiple of T_i. But similarly, we can clear denominators in j, i.e.

$$T_j^b (f|_{U_j}) \in A[T_0, \dots, T_n]$$

for some b. In the intersection, this implies:

$$T_j^b g = T_i^a \left(T_j^b (f|_{U_i \cap U_j}) \right)$$

which shows that $T_j^b g$ is a multiple of T_i^a in $A[T_0, \dots, T_n]$. By uniqueness of polynomial representation, we must have $a = 0$ since T_i divides neither T_j nor g. But this means that $f|_{U_i} \in A$. Since i is arbitrary, this shows that $f \in A$, and since f was arbitrary, this gives that $\mathscr{O}_X(X) = A$.

Now, note that if $n = 0$, then $\mathbb{P}^n_A = \operatorname{Spec} A$ is affine. Conversely, suppose that $\mathbb{P}^n_A$ is affine, so it is isomorphic to $\operatorname{Spec} B$ for some ring B. But then $A = \mathscr{O}_X(X) \cong \mathscr{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) = B$, so we have $\mathbb{P}^n_A \cong \operatorname{Spec} A$. We want to show that this implies $n = 0$. One approach (using material from later sections) is to reduce to the case for fields. Namely, choosing an arbitrary maximal ideal $\mathfrak{m}$ of A (i.e. a closed point) with residue field $k = A/\mathfrak{m}$ gives the following fibre product:

$$\begin{array}{ccc} \mathbb{P}^n_k & \longrightarrow & \mathbb{P}^n_A \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

where the right edge is the claimed isomorphism. But then the left arrow would also be an isomorphism, which is easily disproved by counting. That is, an isomorphism of schemes induces in particular a homeomorphism of topological spaces, but $\operatorname{Spec} k$ consists of a single point, while $\mathbb{P}^n_k$ has at least two points for $n > 0$: both the zero ideal and the ideal generated by T_0 are homogeneous, prime, and do not contain the irrelevant ideal. So, we cannot have $n > 0$ and we conclude $n = 0$ as desired. □

Exercise (2.3.11).

Proof. Note that C is a subring of B, so the inclusion induces a map $i : \operatorname{Spec} B \to \operatorname{Spec} C$. I claim that this restricts to a map $\operatorname{Proj} B \to \operatorname{Proj} C$. Note that if $P \in \operatorname{Proj} B$, then it is a homogeneous prime ideal and $i(P) = P \cap C$ is also homogeneous, for if $f \in P \cap C$, then each nonzero homogeneous component of f must have degree divisible by e by virtue of f being in C, and each such component is therefore in $P \cap C$ as well.

We would now like to also demonstrate that $P \cap C$ does not contain C_+. □

Exercise (2.3.12).

Proof. □

Exercise (2.3.13).

Proof. □

Exercise (2.3.14).

Proof. Note that we've already seen that affine schemes are quasi-compact (exercise 2.1.6). A topological space that is the finite union of quasi-compact subspaces is itself clearly quasi-compact, so one direction is complete: a scheme that is a finite union of affine schemes is quasi-compact. Conversely, suppose X is a quasi-compact scheme. By definition, X has an open covering by affine schemes, and by quasi-compactness, this can be reduced to a finite subcover. So, the claim is shown. □

Exercise (2.3.15).

Proof. Since X is quasi-compact, we can write it as a finite union of affine schemes, say $X = \bigcup_{i=1}^n U_i$ for $U_i = \operatorname{Spec} B_i$. Then, the compositions $U_i \to X \to \operatorname{Spec} A$ induce maps $\varphi_i : A \to B_i$ for each i; in fact, since the map $X \to \operatorname{Spec} A$ is induced by id_A, we have that φ_i is just the restriction map ρ_{X,U_i}. We wish to show that $f(X)$ is dense in $\operatorname{Spec} A$, i.e. we want to compute the closure of $f(X)$. By (exercise 2.3.3) we know that $\overline{f(U_i)} = V(\ker \varphi_i)$, and so:

$$\overline{f(X)} = \overline{f\left(\bigcup_{i=1}^n U_i\right)} = \overline{\bigcup_{i=1}^n f(U_i)} = \bigcup_{i=1}^n \overline{f(U_i)} = \bigcup_{i=1}^n V(\ker \varphi_i) = V\left(\bigcap_{i=1}^n \ker \varphi_i\right) = V(0) = \operatorname{Spec} A$$

We've used multiple facts here. Trivially, we've used that images commute with unions and that closures commute with finite unions. Less trivially, we've noted that $\bigcap_{i=1}^n \ker \varphi_i = 0$. This is in fact just the sheaf condition. For if $g \in \bigcap_{i=1}^n \ker \varphi_i$, then $g|_{U_i} = \varphi_i(g) = 0$, so g is zero on an open cover, and hence globally zero. □

Exercise (2.3.16).

Proof. One direction is trivial. If every affine open subscheme of X is Noetherian, then given a point of X, pick any of its affine open neighborhoods; by assumption this is Noetherian, and so X is locally Noetherian. Now suppose that X is locally Noetherian, and let $U = \operatorname{Spec} A$ be an affine open subscheme. Since X is locally Noetherian, each $x \in U$ has a Noetherian neighborhood of X. Since open subschemes of Noetherian schemes are Noetherian, we may further assume that each x has an open Noetherian neighborhood of U. But affine schemes are quasi-compact, so we can cover U by finitely of these, and hence U is Noetherian, as claimed. □

Exercise (2.3.17).

Proof. Suppose f is a closed immersion. Then, topologically, identifying X with its image, we get that $X \subseteq Y$ is a closed subset. If $U \subseteq Y$ is an affine open subset, then under this identification, the preimage of U under f is $X \cap U$. This is a closed subset of U, and since U is quasi-compact, so is $X \cap U$. This shows that f is quasi-compact.

Suppose f is an open immersion and that Y is locally Noetherian. Making similar identifications as above, we have that $X \subseteq Y$ is open, U is affine, and we want to show that $X \cap U$ is quasi-compact. Since Y is locally Noetherian, the previous exercise shows that U is Noetherian, and so $X \cap U$ is an open subscheme of a Noetherian scheme, and hence itself Noetherian. But this means it is the finite union of affine open subschemes–which are the Specs of Noetherian rings, but we do not use this–and hence quasi-compact, as desired.

Now assume f is quasi-compact. Let $\{U_i\}$ denote an affine open cover of Y. Then, note that $Z = V(\mathscr{I})$ can be computed locally:

$$Z = V(\mathscr{I}) = \{y \in Y \mid \mathscr{I}_y \neq \mathscr{O}_{Y,y}\} = \bigcup_i \{y \in U_i \mid (\mathscr{I}|_{U_i})_y \neq \mathscr{O}_{U_i,y} = \bigcup_i V(\mathscr{I}|_{U_i})\}$$

So, to show that Z is a scheme, it suffices to show that each $V(\mathscr{I}_{U_i})$ is a scheme. In other words, replacing symbols, it suffices to show the claim when Y is affine. Note that we may continue to assume f is quasi-compact since each U_i is open.

Now, with the assumption that $Y = \operatorname{Spec} A$ is affine, we have that f is determined by the map $\varphi : A \to \mathscr{O}_X(X)$. In particular, for a prime $\mathfrak{p} \in \operatorname{Spec} A$,

$$\mathfrak{p} \in Z \iff \mathscr{I}_\mathfrak{p} \neq \mathscr{O}_{Y,\mathfrak{p}} \iff (\ker \varphi)_\mathfrak{p} \neq A_\mathfrak{p} \iff \ker \varphi \subseteq \mathfrak{p} \iff \mathfrak{p} \in V(\ker \varphi)$$

So, $Z = V(\ker \varphi)$ (as topological spaces). In fact, as locally ringed spaces, $Z \cong \operatorname{Spec}(A/\ker \varphi)$, since the canonical map $\operatorname{Spec}(A/\ker \varphi) \to \operatorname{Spec} A = Y$ has image Z and the two agree on principal open subsets. Hence Z is a scheme.

Part d

Part e □

Exercise (2.3.18).

Proof. Since X is an affine variety over k, we can write $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I$ for some ideal I. Let $f_1, \ldots, f_m$ be a set of generators for I. Then, in the ring $k[x_0, x_1, \ldots, x_n]$ obtained by adjoining one additional variable, we can homogenize each f_i to a polynomial g_i by multiplying monomials of lower than maximal degree by an appropriate power of x_0. That is, g_i is homogeneous, $g_i|_{x_0=1} = f_i$, and $x_0 \nmid g_i$.

Let $J = (g_1, \dots, g_m)$, $B = k[x_0, \dots, x_n]/J)$, and $\overline{X} = \operatorname{Proj} B$. I claim this is the desired space. From lemma 3.41, $\overline{X}$ is a projective variety over k with support $V_+(J) \subseteq \mathbb{P}^n_k$. Further, we have

$$D_+(x_0) \cong \operatorname{Spec} B_{(x_0)} = \operatorname{Spec}(k[x_0, \dots, x_n]/J)_{(x_0)} = \operatorname{Spec} k[x_1, \dots, x_n]/I$$

where the last claim follows from an isomorphism of rings. Namely, we have a map $k[x_1, \dots, x_n] \to (k[x_0, \dots, x_n]/J)_{(x_0)}$ by mapping each x_i to $x_0^{-1}x_i$. Under this map, each f_i maps to $\frac{g_i}{x_0^{\deg g_i}} = 0$ in the quotient ring, and so this factors through a map from $k[x_1, \dots, x_n]/I$. The inverse of this map can be similarly defined by evaluating an element of $(k[x_0, \dots, x_n]/J)_{(x_0)}$ at $x_0 = 1$. Under this evaluation, each $g_i/(x_0^{\deg g_i})$ maps to f_i, and so the map is well-defined to $k[x_1, \dots, x_n]/I$. This establishes the desired isomorphism. □

Exercise (2.3.19).

Proof. Let U be an open subset of $\operatorname{Spec} \mathscr{O}_K$. Then $U = \operatorname{Spec} \mathscr{O}_K \setminus V(I)$ for some ideal I of $\mathscr{O}_K$. Since the class group of K has finite order, we know that $I^k = (f)$ is principal for some integer $k > 0$. I claim that $U = D(f)$. Indeed, if $\mathfrak{p} \in \operatorname{Spec} \mathscr{O}_K$, then $\mathfrak{p} \in U$ iff $\mathfrak{p} \notin V(I)$ iff $\mathfrak{p} \not\supseteq I$. Since $\mathscr{O}_K$ is a Dedekind domain, this is equivalent to $\mathfrak{p} \nmid I$, which is true iff $\mathfrak{p} \nmid I^k = (f)$ iff $\mathfrak{p} \in D(f)$. This shows the claim.

From this it is clear that every open subscheme is affine, as $U \cong \operatorname{Spec}(\mathscr{O}_K)_f$. □

Exercise (2.3.20).

Proof. Notice that each $\sigma \in G$ is an invertible map $A \to A$. Let $i : G \to G$ denote the inversion map for notational coherence. Then, taking preimages in the usual way, $i(\sigma)$ induces a map $\operatorname{Spec} A \to \operatorname{Spec} A$. Setting $\sigma \cdot x$ to be the image of x under this induced map is a group action, for if $x \in \operatorname{Spec} A$ and $\sigma, \tau \in G$, then

$$(\sigma \circ \tau) \cdot x = i(\sigma \circ \tau)^{-1}(x) = [i(\tau) \circ i(\sigma)]^{-1}(x) = i(\sigma)^{-1}(i(\tau)^{-1}(x)) = i(\sigma)^{-1}(\tau \cdot x) = \sigma \cdot (\tau \cdot x)$$

as desired. That inverses and the identity behave as expected is clear. Note that taking both inverses amounts to just taking $\sigma \cdot x = \sigma(x)$, the image of x under the function σ.

Note that the map $p : \operatorname{Spec} A \to \operatorname{Spec} A^G$ is given by $p(x) = x \cap A^G$ since we have an inclusion of rings. So, first suppose that $x_1, x_2 \in \operatorname{Spec} A$ are such that there exists $\sigma \in G$ with $\sigma(x_1) = x_2$. Then, explicitly, we have that

$$a \in p(x_1) \iff a \in x_1 \text{ and } a \in A^G \iff \sigma(a) \in \sigma(x_1) \text{ and } a \in A^G \iff a \in x_2 \text{ and } a \in A^G \iff a \in p(x_2)$$

so $p(x_1) = p(x_2)$.

On the other hand, suppose $p(x_1) = p(x_2)$. From the above, we also have $p(x_1) = p(\sigma(x_1))$ for any σ. So, $\sigma(x_1)$ and x_2 lie over the same prime of A^G. Because A is integral over A^G (see below), it suffices to show that $\sigma(x_1)$ contains x_2 for some σ by incomparability. By prime avoidance, it now suffices to show that

$$x_2 \subseteq \bigcup_{\sigma \in G} \sigma(x_1)$$

This is what we will show. Let $a \in x_2$, and consider

$$b = \prod_{\sigma \in G} \sigma(a)$$

Then, one of the multiplicands is $\operatorname{id}(a) = a$, so $b \in x_2$ as well. But clearly each element of G would permute the terms in the product and so leave b fixed. I.e. $b \in A^G$ as well, so $b \in x_2 \cap A^G = p(x_2) = p(x_1) \subseteq x_1$. So, $b \in x_1$, and by primality so is one of the factors, say $\sigma(a) \in x_1$. But then $a \in \sigma^{-1}(x_1)$, which is what we wished to show. This completes the argument.

Now we show explicitly that A is integral over A^G. Let $a \in A$, and consider

$$P(T) = \prod_{\sigma \in G} (T - \sigma(a))$$

This is a monic polynomial; it has a as a root, since one of the factors is $T - \operatorname{id}(a) = T - a$; and the factors are permuted by elements of G, so $P \in A[T]$ is fixed by G. In other words, each coefficient is in A^G, and so we've constructed an integral relation for a over A^G as desired. Since integral extensions satisfy lying over, this also shows that p is surjective.

We show that $p(D(a)) = \bigcup_i D(b_i)$ directly. Suppose $x \in p(D(a))$. Then $x = y \cap A^G$ for some $y \in D(a)$, i.e. some prime y that does not contain a. Suppose, for contradiction, that $b_i \in y$ for each i. Then,

$$a^d = -\sum_i b_i a^i \in y$$

and by primality, we would conclude that $a \in y$, contrary to assumption. So, instead, there is some i for which $b_i \notin y$. Then $b_i \notin x$ since $x \subseteq y$, and so $x \in D(b_i)$. This shows one containment.

For the reverse, let $x \in \bigcup_i D(b_i)$, so there is some i for which $b_i \notin x$. Let y be such that $p(y) = x$, and suppose for contradiction that $\sigma(a) \in y$ for all σ. Then we can write b as a symmetric polynomial in these elements, and so $b_i \in y$ and of course $b_i \in A^G$, so $b_i \in A^G \cap y = p(y) = x$, contrary to assumption. So, instead, there is some σ for which $\sigma(a) \notin y$, i.e. $a \notin \sigma^{-1}(y)$. But then $\sigma^{-1}(y) \in D(a)$, and so $x = p(y) = p(\sigma^{-1}(y)) \in p(D(a))$, as desired. We've shown now that the image under p of a principal open set is open (as a union of principal open sets), and since these form a base for the topology, this shows that p is open.

For $b \in A^G$, we have that

$$y \in p^{-1}(D(b)) \iff b \notin p(y) \iff b \notin y \iff y \in D(bA)$$

so that $p^{-1}(D(b)) = D(bA)$. Only the middle biconditional is non-obvious, so to spell it out, note that if $b \in p(y)$ then clearly $b \in y$ since $p(y) \subseteq y$. Conversely, if $b \in y$, then since $b \in A^G$, we get $b \in y \cap A^G = p(y)$, as claimed. The equality $(A^G)_b = (A_b)^G$ is immediate, since for $b \in A^G$, a fraction in A_b is fixed by G iff the numerator is fixed by G.

To see that G acts on the scheme $p^{-1}(V)$, note that we've already demonstrated that it acts on $\operatorname{Spec} A$ as a scheme, and so it suffices to show that for each $\sigma \in G$, the morphism of schemes $\sigma|_{p^{-1}(V)} : p^{-1}(V) \to \operatorname{Spec} A$ has image in the open subscheme $p^{-1}(V)$. But this is immediate; if $\mathfrak{p}$ is a prime lying over a prime of A^G, then each conjugate of $\mathfrak{p}$ lies over the same prime, and so is also in $p^{-1}(V)$.

Finally, we wish to show the two stated rings are the same. If $V = D(b)$ is a principal open subset, we are done, since:

$$\mathscr{O}_{\operatorname{Spec} A}(p^{-1}(D(b)))^G = \mathscr{O}_{\operatorname{Spec} A}(D(bA))^G = (A_b)^G = (A^G)_b = \mathscr{O}_{\operatorname{Spec} A^G}(D(b))$$

Otherwise, we're still done since $V = \bigcup_i D(b_i)$ for some collection of principal opens, and we obtain the result via gluing. ☐

Exercise (2.3.21).

Proof. For the first claim, let p be as in the previous exercise. We showed already that for $x \in \operatorname{Spec} A$ and $\sigma \in G$ that $p(\sigma(x)) = p(x)$, so $p = p \circ \sigma$ as desired. Now suppose $f : \operatorname{Spec} A \to Z$ is any morphism of schemes with $f \circ \sigma = f$ for all $\sigma \in G$. Let $U = \operatorname{Spec} B \subseteq Z$ be an open affine neighborhood. Then restricting f to $f^{-1}(U)$ gives a map $f : f^{-1}(U) \to U$ which is determined by the ring homomorphism $f^\# : B \to \mathscr{O}_{\operatorname{Spec} A}(f^{-1}(U))$. Note that for any $\sigma \in G$, we have $\sigma \circ f^\# = f^\#$ by the corresponding property on f. So, the image of $f^\#$ lies in the subring $\mathscr{O}_{\operatorname{Spec} A}(f^{-1}(U))^G$.

We would like to identify this with the sheaf of A^G on an open subset, and we could do this if $f^{-1}(U) = p^{-1}(V)$ for some open V. Clearly $f^{-1}(U) \subseteq p^{-1}(p(f^{-1}(U)))$, and since p is an open map, $p(f^{-1}(U))$ is indeed open: so we only need equality. For this, note that if $y \in p^{-1}(p(f^{-1}(U)))$, then $p(y) \in p(f^{-1}(U))$, so there is some $z \in f^{-1}(U)$ with $p(y) = p(z)$. From the previous, this means that there is some $\sigma \in G$ with $\sigma(z) = y$. But then $f(y) = f(\sigma(z)) = f(z) \in U$, so $y \in f^{-1}(U)$ as desired.

In other words, we now have a ring homomorphism

$$f^\# : B \to \mathscr{O}_{\operatorname{Spec} A}(p^{-1}(p(f^{-1}(U))))^G = \mathscr{O}_{\operatorname{Spec} A^G}(p(f^{-1}(U)))$$

This defines a morphism of schemes $g_U : \operatorname{Spec} A^G|_{p(f^{-1}(U))} \to U \hookrightarrow Z$ satisfying $g_U \circ \sigma = g_U$ since the corresponding statement is true for $g_U^\#$. Note that as U ranges over all open affines in Z, $f^{-1}(U)$ defines an open cover of $\operatorname{Spec} A$, and so $p(f^{-1}(U))$ defines an open cover of $\operatorname{Spec} A^G$ since p is open and surjective. So, we've defined morphisms on an open cover, and as long as they agree on overlaps, we can glue them to a morphism $g : \operatorname{Spec} A^G \to Z$. This is essentially automatic, since each morphism comes solely from restricting f. But then since each g_U is G-equivariant, so is G, as can be checked locally.

The argument here is essentially the same. If U is an open subscheme of $\operatorname{Spec} A$ that is G-equivariant, then the map $p|_U : U \to p(U)$ is also G-equivariant, and so satisfies the first part of the universal property for the quotient. Then, if $f : U \to Z$ is also G-equivariant, then we can similarly pull back open subsets of Z to U and push them forward via p to define a map $p(U) \to Z$ via gluing. This argument would then show that $p(U)$ is the quotient U/G.

This also follows from gluing the results of the previous exercise. However, in this case, there is no clear existing candidate for the target scheme X/G, so we must construct the scheme itself via gluing (Lemma 3.33).

Let $\{U_i\}$ denote the collection defined in the problem; that is, for each $x \in X$, there is an affine neighborhood of x that is stable under G, and we let $\{U_i\}$ denote this collection of affine neighborhoods. Each is affine, so we have a corresponding collection of rings: $U_i = \operatorname{Spec} A_i$. Now, define $X_i = \operatorname{Spec}(A_i^G) = (\operatorname{Spec} A_i)/G$, for each i, and $p_i : U_i \to X_i$ the projection we've been considering. For a pair of indices i, j, define $X_{ij} = p_i(U_i \cap U_j) \subseteq X_i$, and let $f_{ij} : X_{ij} \to X_{ji}$ be given by $f_{ij} = p_j \circ p_i^{-1}$.

Foremost, we need to see that each f_{ij} is well-defined. In particular, we're not claiming that p_i is invertible, even when restricted to $p_i(U_i \cap U_j)$, but rather that p_j is constant on this preimage. Indeed, this is clear, for if $p_i(x) = p_i(y)$, then there is some $\sigma \in G$ with $\sigma(x) = y$, whence $p_j(x) = p_j(y)$ as well. Then, it is also the case that the collections $\{X_{ij}\}, \{f_{ij}\}$ satisfy the conditions of Lemma 3.33. So, we define X/G to be the ($\mathbb{Z}$-)scheme guaranteed by that lemma, and $g_i : X_i \to X/G$ the guaranteed maps.

Finally, we would like to justify the name X/G. Let $p : X \to X/G$ be given by $p(x) = g_i(p_i(x))$ for any i such that $x \in U_i$. This is well-defined, for if $x \in U_j$ also, then

$$g_i(p_i(x)) = g_j(f_{ij}(p_i(x))) = g_j(p_j(x))$$

Now, if $\sigma \in G$, then for any $x \in U_i$, we have $\sigma(x) \in U_i$ also since each U_i is G-equivariant, so:

$$p(\sigma(x)) = g_i(p_i(\sigma(x))) = g_i(p_i(x)) = p(x)$$

since $p_i \circ \sigma = p_i$ for each i. So, $p \circ \sigma = p$ as desired. Lastly, suppose we have a morphism $f : X \to Z$ with $f \circ \sigma = f$ for all $\sigma \in G$. Then for $x \in X$, define $h : X/G \to Z$ by $h(p(x)) = f(x)$.

Again, we need to see that this is well-defined. Suppose $p(x) = p(z)$. Then, if $x \in U_i$, $z \in U_i$ also by invariance of each U_i. So, we have

$$g_i(p_i(x)) = p(x) = p(z) = g_i(p_i(z))$$

Since each g_i is an open immersion, this gives $p_i(x) = p_i(z)$. Then, note that for each $\sigma \in G$, that $f|_{U_i} \circ \sigma = f|_{U_i}$, so this restriction factors through the quotient. That, is there is a map $h_i : U_i/G = p(U_i) \to Z$ with $f|_{U_i} = h_i \circ p_i$. Overall:

$$f(x) = f|_{U_i}(x) = h_i(p_i(x)) = h_i(p_i(z)) = f|_{U_i}(z) = f(z)$$

and so h, as above, is well-defined. But this completes the argument, for we've defined h such that $f = h \circ p$, i.e. we've shown that f factors through the quotient map, so X/G is indeed the quotient. □

Exercise (2.3.22).

Proof. It is clear that n is an automorphism of $k[T]$ for each n, with inverse $-n$, and thus of $\mathbb{A}_k^1$. Now, suppose that U is an open subset of $\mathbb{A}_k^1$. Then the complement of U is closed, so it is of the form $V(I)$ for some ideal I, but since $k[T]$ is a PID, we can write $I = (f)$ for some polynomial f. Then $V(I)$ is precisely the set of prime factors of f, with U being its complement. If $f \in k$ is a unit, then it has no prime divisors and $U = \mathbb{A}_k^1$. If $f = 0$, then every prime divides it, and so $U = \emptyset$.

Otherwise, $V(f)$ is nonempty and finite; in particular, it contains some prime ideal $(g(T))$. But if $(g(T+n)) \notin V(f)$, then $(g(T+n)) \in U$, and so $(g(T+n-n)) = (g(T)) \in U$, contrary to assumption. So, $V(f)$ contains $(g(T+n))$ for every n. Since we are in characteristic zero, $(g(T+n)) = (g(T+m))$ iff $n = m$, and so we have our contradiction, since $V(f)$ is supposed to be finite, but we have an injection $\mathbb{Z} \hookrightarrow V(f)$ given by $n \mapsto (g(T+n))$.

Now, note that the ring homomorphism $\varphi : k \hookrightarrow k[T]$ induces a map $p : \mathbb{A}_k^1 \to \operatorname{Spec} k$. Further, the ring map is $\mathbb{Z}$-invariant, since $n \cdot \varphi(a) = \varphi(a)$, since $\varphi(a)$ is a constant and has no T-terms. So, $p \circ n = p$ for any $n \in \mathbb{Z}$. Further, p has the universal property of the quotient map of schemes. Indeed, let Z be any scheme and $f : \mathbb{A}_k^1 \to Z$ a morphism with $f \circ n = f$ for all $n \in \mathbb{Z}$. Then, pick any z in the image of f, and pick an affine neighborhood U of z in Z. Then $f^{-1}(U)$ is a nonempty open subset of $\mathbb{A}_k^1$, and I claim it is $\mathbb{Z}$-stable. For if $x \in f^{-1}(U)$, then $f(n(x)) = f(x) \in U$, and so $n(x) \in f^{-1}(U)$ as well. But by the previous claim, this forces $f^{-1}(U) = \mathbb{A}_k^1$ to be the whole space.

In other words, f factors through a map $f : \mathbb{A}_k^1 \to U = \operatorname{Spec} A$ (which I'm also denoting by f). But this is precisely determined by a map $f^\# : A \to k[T]$, and the $\mathbb{Z}$-invariance implies that $f^\#(a)(T+n) = f^\#(a)(T)$ for all $n \in \mathbb{Z}$ and $a \in A$. In other words, each image is a constant polynomial, and so $f^\#$ has image in k, whence f factors through p. But this is precisely what we wished to show.

On the other hand, from exercise 2.2.14, we know that the quotient $\mathbb{A}_k^1/\mathbb{Z}$ of ringed topological spaces exists and has underlying topological space homeomorphic to the quotient topological space $\mathbb{A}_k^1/\mathbb{Z}$. In other words, choosing two elements $a, b \in k$ such that $a - b \notin \mathbb{Z}$, we have that the ideals $(T-a)$ and $(T-b)$ map to different points in the ringed topological space quotient; in particular the ringed topological space has more than one point. But the topological space of $\operatorname{Spec} k$ is a single point, and so the two are not the same as ringed topological spaces (or even as topological spaces). □

Exercise (2.4.1).

Proof. Note that an affine scheme $\operatorname{Spec} A$ is reduced (irreducible) iff A is reduced (has a unique minimal prime, respectively). For each of the following, write $P = Q_1^{e_1} \cdots Q_m^{e_m}$ for Q_i distinct primes, which can be done since $k[T_1, \ldots, T_n]$ is a UFD. Then, by CRT, we can write

$$k[T_1, \ldots, T_n] \cong \bigoplus_{i=1}^{m} k[T_1, \ldots, T_n]/(Q_i^{e_i})$$

Let $A = k[T_1, \ldots, T_n]/(P)$ for what follows.

Now, we have $\operatorname{Spec} A$ is reduced iff A is reduced. If P is squarefree, then each $e_i = 1$, and the above isomorphism exhibits the A as the direct sum of domains, which is reduced. Conversely, if P is not squarefree, then $e_i \geq 2$ for some i. Then P/Q_i is in $k[T_1, \ldots, T_n]$ but not in (P) and $P^2/Q^2 = (P/Q^2)P \in (P)$, so the image of P/Q is zero in the quotient and A is not reduced.

Second, recall that primes in a direct sum correspond to the (disjoint) union of primes of each summand. So, if $m \geq 2$, then (Q_1) and (Q_2) are distinct minimal primes of A, whence A is not irreducible. Conversely, if $m = 1$, then any prime ideal of A corresponds to a prime of $k[T_1, \ldots, T_n]$ containing $P = Q_1^{e_1}$, and so this prime contains Q_1. Thus, (Q_1) is the unique minimal prime of A and so A is irreducible.

Finally, combining these, $\operatorname{Spec} A$ is integral iff $m = 1$ and $e_1 = 1$, iff $P = Q_1$ is irreducible. □

Exercise (2.4.2).

Proof. Suppose y specializes to x. Then, choose an open affine $U = \operatorname{Spec} A$ containing x. We must have $y \in U$ as well, else we would have $x \in \overline{\{y\}} \subseteq X \setminus U$. Then we recall that the map $\operatorname{Spec} \mathscr{O}_{X,x} \to X$ is given by choosing any open affine containing x and factoring the localization map. In particular, for this U, we get that the image of $\operatorname{Spec} \mathscr{O}_{X,x} \to X$ contains the image of the morphism induced by the localization map $A \to A_x$, which is all primes of A contained in x. But the statement that x is a specialization of y is precisely that $x \supseteq y$ as primes of A, and so y is in the image.

Conversely, suppose y is in the image of $\operatorname{Spec} \mathscr{O}_{X,x} \to X$. Then, if $U = \operatorname{Spec} A$ is any open affine containing x, we know that the map factors through the open immersion $U \to X$, and so $y \in U$. But then the map is given by localization of A at x, and so we must have $y \subseteq x$ as ideals of U, whence y specializes to x. □

Exercise (2.4.3).

Proof. Note that the principal open subscheme $D(t)$ is isomorphic to $\operatorname{Spec}((\mathscr{O}_K[T])_t) = \operatorname{Spec}(K[T])$ since inverting the uniformizer gives the field of fractions for a DVR.

Geometrically, recall that $\operatorname{Spec} \mathscr{O}_K$ is a two-point space, with the open (generic) point, and the closed point. Then, $\operatorname{Spec} \mathscr{O}_K[T]$ is a pair of lines, an "open" line corresponding to $\operatorname{Spec} K[T]$ and a closed line. The closed point (t, T) is the origin of the closed line, and we are asking for closed points of the open line that specialize to the origin. From this model, we should expect only the origin of the open line will specialize to the origin, i.e. the point corresponding to the ideal (T) in $K[T]$.

Now, let's prove this algebraically. Let $P \in \operatorname{Spec} K[T]$ be a closed point that specializes to (t, T) in $\operatorname{Spec} \mathscr{O}_K[T]$. In other words, we have $P \cap \mathscr{O}_K[T] \subseteq (t, T)$. Now, since $K[T]$ is a PID, we can write $P = (f)$ for some f, and since P is maximal, it is nonzero, and so f is irreducible. Clearing denominators, we have that $af \in P \cap \mathscr{O}_K[T]$ for some $a \in \mathscr{O}_K^\times$. Now, if f has a nonzero constant term $b \in K$, then $ab \in (t, T)$ since $af, T \in (t, T)$. But ab is a unit, contradicting that (t, T) is proper. I.e. we must have that $f(0) = 0$, but then $T \mid f$. By irreducibility, we have that $(f) = (T)$, i.e. that P is the origin in $\operatorname{Spec} K[T]$ as claimed. □

Exercise (2.4.4).

Proof. We associate to X a graph $G = (V, E)$ defined as follows: for each irreducible component X_i, we define a vertex $v_i \in V$, and if $X_i \cap X_j \neq \emptyset$, then $(v_i, v_j) \in E$. Now, the claim we are trying to show is equivalent to saying that X is connected if and only if G is a connected graph.

First, suppose that X is connected. To show that G is connected, it suffices to show the following property: if $V = S \sqcup T$ is partitioned into any two nonempty subsets, then there is an edge $(s, t) \in E$ with $s \in S$ and $t \in T$. To see this, let S, T be as stated, and let

$$W = \bigcup_{i \in S} X_i \text{ and } Z = \bigcup_{i \in T} X_i$$

Since these are both finite nonempty unions and each X_i is closed, we have that W and Z are nonempty and closed. But also we have that $X = W \cup Z$, and so by connectedness, these cannot be disjoint, i.e. $W \cap Z \neq \emptyset$. Any point in this intersection lies in some X_i for $i \in S$ and some X_j for $j \in T$. But then $X_i \cap X_j \neq \emptyset$ and $(i, j) \in E$ as desired.

Conversely, suppose that G is connected. Write $X = W \sqcup Z$ for two disjoint closed subsets of X; we wish to show that one of them is empty. For each i, note that $X_i = (X_i \cap W) \cup (X_i \cap Z)$, and so by irreducibility, either $X_i = X_i \cap W$ or

$X_i = X_i \cap Z$. In other words, each irreducible component is contained in at least one of W or Z. Let $S = \{i : X_i \subseteq W\}$ and $T = \{i : X_i \subseteq Z\}$. We've shown that $V = S \cup T$. In fact, S and T are disjoint, for if $i \in S \cap T$, then $X_i \subseteq W \cap Z = \emptyset$. Suppose both S and T are nonempty. From the above equivalent condition for connectedness of a graph, we have that there is some edge (s, t) with $s \in S$ and $t \in T$. In other words, we have $X_s \cap X_t \neq \emptyset$. But we have $X_s \cap X_t \subseteq W \cap Z = \emptyset$. This is a contradiction, and so we cannot have both S and T nonempty; WLOG assume $S = \emptyset$. Then $T = V$, so $X_i \subseteq Z$ for all i, and so $Z = X$ and $W = \emptyset$, which is what we wished to show.

Now, suppose X is integral. We've already noted that this implies that X is integral at x for each $x \in X$. Further, we have that X is irreducible, so in the language above, V is a singleton. The graph on 1 vertex is certainly connected, and so X is connected.

Conversely, suppose X is integral at each of its points and connected. Then X is clearly reduced at each of its points since domains are reduced, and so it suffices to show that X is irreducible. But note that being integral at each of its points implies that each $x \in X$ is contained in a unique irreducible component, and so in the language of graphs above, $E = \emptyset$. That is, $X_i \cap X_j = \emptyset$ for all i, j. But now, we have a connected graph with no edges, and so we must have $|V| \leq 1$, i.e. there is at most one irreducible component. This component must be all of X, and so X is irreducible, and hence integral as desired. □

Exercise (2.4.5).

Proof. Note that the connected component of X containing a point $x \in X$ is the union of all connected subsets of X containing x. Thus, if Z is any irreducible subset of X, then Z is connected, so Z is contained in the connected component of any $x \in Z$.

Now, suppose X is locally Noetherian, and let U be a connected component. We wish to show U is open, so let $x \in U$. Since X is locally Noetherian, x has an open neighborhood that is the finite union of affine neighborhoods of Noetherian rings. Then x is contained in one such neighborhood, say $x \in V = \operatorname{Spec} A$ where A is Noetherian. It now suffices to exhibit a connected open subset of V containing x.

For this, note that there are only finitely many irreducible components in V. We induct on the number of such components that do not contain x. If there are none, then each irreducible component of V overlaps (at x), and so by the previous problem, V itself is connected and is of course open in V. So, suppose there is at least one irreducible component Z not containing x. Then the complement of Z is open, so it is covered by principal open sets; choose $f \in A$ so that $x \in D(f) \subseteq V \setminus Z$. The irreducible components of $D(f)$ are given by intersecting irreducible components of V with $D(f)$; in particular Z does not appear. So, in $D(f)$, there are strictly fewer irreducible components that do not contain x. Finally, $D(f) \cong \operatorname{Spec} A_f$ is the spectrum of a Noetherian ring, so induction applies and gives an open connected subset $W \subseteq D(f)$ containing x. But $D(f)$ is open in V, and so W is open in V as desired.

Lastly, note that each irreducible component is contained in some connected component, and each connected component contains at least one irreducible component. So, there is a surjection from the set of irreducible components to the set of connected components. When X is Noetherian, there are only finitely many of the former and so only finitely many of the latter. □

Exercise (2.4.6).

Proof. Note that it suffices to prove the equivalence of (i) and (ii). Indeed, suppose we have done so, let X be a scheme, let $A = \operatorname{Spec} \mathscr{O}_X(X)$, and apply the proven equivalence to the scheme $\operatorname{Spec} A$. Then we conclude $\operatorname{Spec} \mathscr{O}_X(X) = \operatorname{Spec} A$ is connected iff $\mathscr{O}_{\operatorname{Spec}(A)}(\operatorname{Spec} A) = A = \mathscr{O}_X(X)$ has no nontrivial idempotents, which is precisely (ii) iff (iii).

So now, suppose first that X is disconnected. Then we can write $X = U \cup V$ for disjoint open sets U and V. The sections $1 \in \mathscr{O}_X(U)$ and $0 \in \mathscr{O}_X(V)$ trivially agree on the overlap $U \cap V = \emptyset$. So we can glue them to a section $f \in \mathscr{O}_X(X)$. Then $f \neq 0$ since the restriction to U is nonzero, and similarly $f \neq 1$. But $f^2|_U = (f|_U)^2 = 1^2 = 1$ and $f^2|_V = (f|_V)^2 = 0^2 = 0$, so $f^2 = f$ shows that $\mathscr{O}_X(X)$ has a nontrivial idempotent.

Conversely, suppose that $e \in \mathscr{O}_X(X)$ is a nontrivial idempotent. Then consider the two open subsets X_e and X_{1-e}. Since $e(1-e) = e - e^2 = e - e = 0$, we cannot have e_x and $(1-e)_x$ both be units in any stalk $\mathscr{O}_{X,x}$. Hence X_e and X_{1-e} are disjoint. On the other hand, this same equation shows that one of $e, 1-e$ must be contained in the unique maximal ideal of $\mathscr{O}_{X,x}$. Finally, $e_x + (1-e)_x = 1$ is not contained in this maximal ideal, and so exactly one of them is contained in the maximal ideal. Thus one of them is a unit, whence x is contained in at least one of X_e and X_{1-e}. Thus we've written X as the disjoint union of open subsets, so X is disconnected.

If $(A, \mathfrak{m})$ is local, then any closed subset of $\operatorname{Spec} A$ is of the form $V(I)$, and $I \subseteq \mathfrak{m}$, so $\mathfrak{m} \in V(I)$. Thus there are no disjoint closed subsets and so $\operatorname{Spec} A$ is connected.

Notice that under these assumptions, U is clearly open and $X \setminus U$ is the union of all other connected components and so is also open. Thus, the uniqueness of e is obvious, since we've specified its restriction to an open cover, and existence follows by gluing.

The next claim is false in characteristic 2. For example, in $\mathrm{Spec}(\mathbb{F}_2 \times \mathbb{F}_2)$, $e = (1,0)$ is the claimed section, but it is not indecomposable, as it is the sum of the idempotents $(1,1)$ and $(0,1)$. Outside of characteristic 2, however, we are okay. Indeed, write e as above as the sum of two idempotents $e = f + g$. Then $1 = e|_U = f|_U + g|_U$, but U is connected, so the only idempotents are $0, 1$. Thus, WLOG, $f|_U = 0$ and $g|_U = 1$. For each other connected component V, we get $0 = e|_V = f|_V + g|_V$, and so $f|_V = g|_V = 0$ since again V is connected and $\mathscr{O}_X(V)$ has characteristic $\neq 2$. So, gluing back gives $f = 0$ and $g = e$, and so e is indecomposable.

So, finally, we show that this association is bijective. It's clearly bijective, for if e corresponds to U and f corresponds to V for components $U \neq V$, then $e|_V = 0$ while $f|_V = 1$, so $e \neq f$. So we wish to show it is surjective; let $e \in \mathscr{O}_X(X)$ be any nonzero indecomposable idempotent. Since it is nonzero, it cannot be zero when restricted to each connected component. So, there is some component U with $e|_U \neq 0$. Since U is connected and $e|_U$ is idempotent, we must have $e|_U = 1$. Now, suppose $e|_V \neq 0$ for some other component V. Then $e = (e - f) + f$, where f is the idempotent corresponding to V and $e - f, f$ are nontrivial. This contradicts indecomposability, and so we must have $e|_V = 0$ for each other component, whence e is precisely the idempotent corresponding to U. □

Exercise (2.4.7).

Proof. Let $x \neq y$ be two points of a scheme X. Let $U = \mathrm{Spec}\, A$ be an open affine neighborhood of x. If $y \notin U$, then we're done as we've found an open set containing one point and not the other.

Otherwise $y \in U$, so x, y are distinct prime ideals in A. As sets, we therefore have either $x \not\subseteq y$ or $y \not\subseteq x$. In the first case, $y \notin V(x)$ so $y \in U \setminus V(x)$ while $x \notin U \setminus V(x)$ giving an open set containing y but not x. In the second, $x \in U \setminus V(y)$ while $y \notin U \setminus V(y)$, completing the proof. □

Exercise (2.4.8).

Proof. For points $x, y \in X$, define $x \leq y$ if $y \in \overline{\{x\}}$, i.e. if x specializes to y. First, I claim this is a partial order on X. It is clearly reflexive as $x \in \overline{\{x\}}$. It is also transitive: if $x \in \overline{\{y\}}$ and $y \in \overline{\{z\}}$, then $\overline{\{z\}}$ is a closed set containing y, whence it contains the full closure $\overline{\{y\}}$, and therefore $x \in \overline{\{z\}}$.

Finally, suppose $x \neq y$. Then, from the previous exercise, X is T_0, so WLOG there is some open set U containing x but not y. But then $y \in X \setminus U$, a closed set, and so $\overline{\{y\}} \subseteq X \setminus U$. But x is not in the latter, so it cannot be in the former, i.e. $y \not\leq x$. In other words, $\leq$ is antisymmetric, and therefore a partial order.

Now, let $C \subseteq X$ be a totally ordered subset. Let

$$U = \bigcup_{x \in C} X \setminus \overline{\{x\}}$$

First, suppose that $U = X$. Then we've exhibited an open cover of X, and so by quasi-compactness, we conclude that there are $x_1, \ldots, x_n \in C$ with

$$X = \bigcup_{i=1}^{n} X \setminus \overline{\{x_i\}} \implies \emptyset = \bigcap_{i=1}^{n} \overline{\{x_i\}}$$

Since C is totally ordered, any finite subset has a maximum. WLOG, we may assume $x_1 \geq x_i$ for each i, i.e. $x_1 \in \overline{\{x_i\}}$ for each i, contradicting the fact that the intersection of these closures is empty.

So, we must have that U is proper. Let $z \in X \setminus U$. Then, for all $x \in C$, we have $z \in \overline{\{x\}}$, i.e. $z \geq x$. So z is an upper bound for C. Thus, we've shown that any totally ordered subset of X has an upper bound, i.e. Zorn's lemma applies. Thus, let $y \in X$ be a maximal element. Then if $x \in \overline{\{y\}}$, we get $x \geq y$ and so $x = y$ by maximality. So, we have $\overline{\{y\}} = \{y\}$, which implies that y is a closed point, which is what we wished to show. □

Exercise (2.4.9).

Proof. Let $x \in X$ be a point with $\mathscr{O}_{X,x}$ is reduced. Pick an affine neighborhood $\mathrm{Spec}\, A$ of x, and note that since X is Noetherian, so is A. Think now of x as a prime ideal of A. Then $\mathscr{O}_{X,x} = A_x$, the localization of A at the prime x. Since this is a reduced ring, we conclude that $0A_x = \mathscr{N}(A_x) = \mathscr{N}(A)_x$, where $\mathscr{N}$ denotes the nilradical. Since A is Noetherian, we can write $\mathscr{N}(A) = (f_1, \ldots, f_n)$, and the fact that it localizes to zero means that there exist $a_1, \ldots, a_n \in A \setminus x$ with $a_i f_i = 0$. Define

$$U = \left(\bigcap_{i=1}^{n} D(a_i) \right)$$

This is the finite intersection of open sets, and so is open in $\operatorname{Spec} A$, and so also open in X. I claim $\mathscr{O}_{X,y}$ is reduced for each $y \in U$, which would complete the argument for reducedness. Indeed, let $y \in U$ and note that $\mathscr{O}_{X,y} = A_y$. Such an ideal y then satisfies $a_i \notin y$ for each i, and so $f_i/1 = a_i f_i / a_i = 0$ in A_y. Thus, $\mathscr{N}(A_y) = \mathscr{N}(A)_y = (f_1, \ldots, f_n)_y = 0A_y$, showing that A_y is reduced.

Now, suppose $x \in X$ is such that $\mathscr{O}_{X,x}$ is integral. In other words, $\mathscr{O}_{X,x}$ is reduced and x is contained in a unique irreducible component Z. First, let U be an open neighborhood of x such that $\mathscr{O}_{X,y}$ is reduced for each $y \in U$ as above. Then, note that since X is Noetherian, it has finitely many irreducible components $Z = Z_0, Z_1, \ldots, Z_n$, and we can define:

$$V = U \setminus \left(\bigcup_{i=1}^{n} Z_i \right)$$

Then $x \in V$ since $x \notin Z_i$ for $i \neq 0$, and V is open since it is U minus the finite union of closed sets. Each $y \in V$ also then lies only on Z, and so $\mathscr{O}_{X,y}$ is integral as desired. ☐

Exercise (2.4.10).

Proof. Suppose $f(X) \subseteq V(I)$, and let $g \in f^{\#}(\operatorname{Spec} A)(I)$. Let $U = \operatorname{Spec} B$ be an open affine subset of X and note then that the restriction of $f^{\#}$ is a ring homomorphism $A \to B$, so there is some $h \in I$ with $g|_U = f^{\#}(h)$. Now, if P is a prime in B, then $(f^{\#})^{-1}(P) = f(P) \in V(I)$ and so contains h. So, $g|_U = f^{\#}(h) \in P$. This is true for all P, and so $g|_U$ is contained in the nilradical of B, and so there is some n with $(g|_U)^n = 0$.

Now, since f is quasi-compact, $f^{-1}(\operatorname{Spec} A) = X$ is quasi-compact, and so we can cover X by finitely many affine neighborhoods $U_1, \ldots, U_m$. The argument above shows that there are $n_1, \ldots, n_m$ with $(g|_{U_j})^{n_j} = 0$. But then for $n = \max\{n_1, \ldots, n_m\}$, we have $(g^n)|_{U_j} = (g|_{U_j})^n = 0$, and so $g^n = 0$ on X by gluing. This shows one implication.

Conversely, suppose $f^{\#}(\operatorname{Spec} A)(I)$ is nilpotent. Let $x \in X$ and pick an affine neighborhood $U = \operatorname{Spec} B$ of x. Then, if $g \in I$, we have $f^{\#}(g)$ is nilpotent and so contained in each prime of B, including x, and so $g \in (f^{\#})^{-1}(x) = f(x)$. So $I \subseteq f(x)$ and $f(x) \in V(I)$. Since x was arbitrary, $f(X) \subseteq V(I)$ as desired. ☐

Exercise (2.4.11).

Proof. (ii) $\implies$ (iii). Let $U \subseteq f^{-1}(V)$, and consider the maps:

$$\mathscr{O}_Y(V) \to \mathscr{O}_X(f^{-1}(V)) \to \mathscr{O}_X(U)$$

The first map is injective by assumption, and so it suffices to show the second is injective. But the injection $\mathscr{O}_X(f^{-1}(V)) \to \mathscr{O}_{X,\xi_X}$ factors through this map, so we're done.

(iii) $\implies$ (iv). Let $V = \operatorname{Spec} A \subseteq Y$ be an open affine. Then $f^{-1}(V)$ is open in X and so necessarily contains ξ_X. Choosing an open affine in X containing ξ_X, we can restrict to a principal open set contained in $f^{-1}(V)$, giving us $U = \operatorname{Spec} B \subseteq f^{-1}(V)$. Under these identifications, the injection $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is a map $A \to B$ and the generic points are the zero ideals of A, B. Since the map injects, the preimage of $0B$ is $0A$, so $f(\xi_X) = \xi_Y$.

(iv) $\implies$ (v). Trivial.

(v) $\implies$ (i). This is essentially trivial: $\overline{f(X)} \supseteq \overline{\xi_Y} = Y$.

(i) $\implies$ (iv). Suppose f is dominant. Then

$$Y = \overline{f(X)} = \overline{f(\overline{\{\xi_X\}})} = \overline{f(\{\xi_X\})} = \overline{\{f(\xi_X)\}}$$

so $f(\xi_X)$ is a generic point of Y, but ξ_Y is the unique one.

(iv) $\implies$ (ii). Suppose $f^{\#}$ is not injective, whence it is not injective on stalks, so there is some $x \in X$ such that the map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is not injective. We have the commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{Y,f(x)} & \longrightarrow & \mathscr{O}_{X,x} \\ \downarrow & & \downarrow \\ \mathscr{O}_{Y,\xi_Y} & \longrightarrow & \mathscr{O}_{X,\xi_X} \end{array}$$

and since the vertical arrows are injections, the bottom arrow must also fail to be injective. But this is a map of fields, so this is impossible. ☐

Exercise (2.4.12).

Proof. Let Y be a reduced closed subscheme of $\operatorname{Proj} B$. As a set, Y is closed, so it is of the form $V_+(I)$ for some homogeneous ideal I of B. Without loss of generality, we may assume I is radical, since $\sqrt{I}$ is also homogeneous and $V_+(\sqrt{I}) = V_+(I)$.

Now, note that the map $B \to B/I$ induces a closed immersion $\operatorname{Proj}(B/I) \to \operatorname{Proj}(B)$ since the map respects the irrelevant ideals. But then $\operatorname{Proj}(B/I)$ is a reduced closed subscheme of $\operatorname{Proj}(B)$ that has the same underlying set as Y. By uniqueness, $Y = \operatorname{Proj}(B/I)$. □

Exercise (2.5.1).

Proof. It is clear that $\dim X \geq \sup_i \dim X_i$, since each X_i is contained in X. Now, let $x \in X$, and choose an open neighborhood U of x that intersects only finitely many X_i. Then, since the X_i cover, we have:

$$U = \bigcup_{j=1}^{n} U \cap X_{i_j}$$

for some indices $i_1, \ldots, i_n$. Now, consider a chain of irreducible closed subsets of U:

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m$$

The decomposition above shows:

$$Z_m = \bigcup_{j=1}^{n} Z_m \cap X_{i_j}$$

and irreducibility gives, after possibly rearranging, $Z_m = Z_m \cap X_{i_1}$, i.e. $Z_m \subseteq X_{i_1}$. But then the closure of each Z_i gives a chain of irreducible closed subsets of X_{i_1}, and so $\dim(X_{i_1}) \geq m$. To avoid rearranging, we can note therefore that $\sup_i \dim(X_i) \geq m$. Since this applies for each m, we can take suprema again to get $\dim(U) \leq \sup_i \dim(X_i)$.

Since this is true for a particular U, we therefore also get $\dim_x(X) = \inf_{U \ni x}(\dim(U)) \leq \sup_i \dim(X_i)$, and finally, taking suprema over all $x \in X$ gives $\dim(X) \leq \sup_i \dim(X_i)$, as desired. □

Exercise (2.5.2).

Proof. Note that $\overline{\{x\}}$ is irreducible and closed in X. So, the codimension is the supremum of lengths of strictly ascending chains of irreducible closed subsets between $\overline{\{x\}}$ and X. Choose an open affine $U = \operatorname{Spec} A$ containing x. Then we have a bijection between such chains and chains of irreducible closed subsets of U containing x, given by taking intersections with U for one direction and taking closures in X for the other.

So, considering x as a prime ideal in A, we are considering chains of irreducibles in U containing x, which is precisely chains of primes in A contained in x, which is precisely chains of primes in $A_x = \mathscr{O}_{X,x}$. So, the codimension is exactly $\dim(\mathscr{O}_{X,x})$. □

Exercise (2.5.3).

Proof. If Z contains an irreducible component W, then

$$\operatorname{codim}(Z, X) = \inf_{T \text{ irred comp of } Z} \operatorname{codim}(T, X) \leq \operatorname{codim}(W, X) = 0$$

since W is a maximal irreducible closed subset by assumption. Conversely, if $\operatorname{codim}(Z, X) = 0$, then $\operatorname{codim}(T, X) = 0$ for some irreducible component T of Z. Then T is irreducible and closed in X, so it is contained in an irreducible component W of X. But if $T \neq W$, then the chain $T \subseteq W$ shows $\operatorname{codim}(T, X) \geq 1$, contrary to assumption. So $T = W$ is an irreducible component of X itself.

If we let X be the union of a plane and a line and let Z be the line, then Z is an irreducible component of X, and so has codimension zero, but $\dim(Z) = 1$ and $\dim(X) = 2$. Explicitly, let $A = k[x, y, z]/(xz, yz)$ for any field k, let $X = \operatorname{Spec} A$, and let $Z = V(x, y)$. Clearly Z is closed, and it is irreducible since (x, y) is a prime ideal of A. The codimension is zero as claimed since (x, y) is a minimal prime of A. Further, Z is one-dimensional since $A/(x, y) \cong k[z]$ is one-dimensional. But X is two dimensional, since we have the sequence of primes

$$(z) \subsetneq (y, z) \subsetneq (x, y, z)$$

in A.

Let $A = \mathscr{O}_K$ and note that $A[T]/(tT - 1) \cong A_t \cong \operatorname{Frac}(A)$, and so (f) is maximal. Since $A[T]$ is a domain, X is integral and so clearly irreducible. We have $\dim \mathscr{O}_{X,x} = \operatorname{ht}(f) = 1$, where the former is essentially the definition, and the latter follows from Krull's principal ideal theorem, and the fact that (0) is prime. Also $\dim\{x\} = 0$ since (f) is maximal. Finally, $\dim(X) \geq 2$ since we have the chain of primes $(0) \subsetneq (T) \subsetneq (T, t)$ in $A[T]$. But this shows the claim: $\operatorname{codim}(\{x\}, X) + \dim\{x\} = 0 + 1 < 2 \leq \dim(X)$. □

Exercise (2.5.4).

Proof. The claim is obvious for $r = 1$, since it says that if $I \not\subseteq \mathfrak{p}$, then $I \not\subseteq \mathfrak{p}$.

Now, suppose the claim is known for $r-1$ and $I \not\subseteq \mathfrak{p}_i$ for $i = 1, \dots, r$. If $\mathfrak{p}_r \supseteq \mathfrak{p}_i$ for some $i < r$, then $\bigcup_{j \neq i} \mathfrak{p}_j = \bigcup_j \mathfrak{p}_j$. But then the induction hypothesis gives $I \not\subseteq \bigcup_{j \neq i} \mathfrak{p}_j$ completing the proof.

Otherwise, $\mathfrak{p}_r$ doesn't contain any other $\mathfrak{p}_i$. So, we can choose elements $y_1, \dots, y_{r-1}$ with $y_i \in \mathfrak{p}_i \setminus \mathfrak{p}_r$. Also choose $z \in I \setminus \mathfrak{p}_r$. Then, let

$$y = z \cdot y_1 \cdots y_{r-1}$$

Since $\mathfrak{p}_r$ is prime and none of the multiplicands are in $\mathfrak{p}_r$, we get $y \notin \mathfrak{p}_r$. But also $y \in I\mathfrak{p}_1 \cdots frp_{r-1}$ by construction as suggested. By the induction hypothesis, $I \not\subseteq \bigcup_{i<r} \mathfrak{p}_i$, so we can choose $x \in I \setminus \left(\bigcup_{i<r} \mathfrak{p}_i\right)$ also as suggested.

Finally, we have that $x, x+y \in I$, so we would be done if at least one of these is not in $\bigcup_i \mathfrak{p}_i$. Suppose $x \in \bigcup_i \mathfrak{p}_i$. Then by construction, $x \not\mathfrak{p}_1, \dots, \mathfrak{p}_{r-1}$, so we get $x \in \mathfrak{p}_r$. But then $x + y \in \mathfrak{p}_r \implies y = (x+y) - x \in \mathfrak{p}_r$, which we know is not true. So $x + y \notin \mathfrak{p}_r$. Further, for $i < r$, we have $x + y \not\mathfrak{p}_i$ for a similar reason: if it were, then $x = (x+y) - y \in \mathfrak{p}_i$ as well, which it isn't. So, $x + y$ isn't in any $\mathfrak{p}_i$ as desired. □

Exercise (2.5.5).

Proof. The argument is nearly identical to the previous; we only need to keep track of degrees. We can again assume $\mathfrak{p}_r$ contains no $\mathfrak{p}_i$ for $i < r$. So we can choose elements $y_i \in \mathfrak{p}_i \setminus \mathfrak{p}_r$. For each i, if $\mathfrak{p}_r$ contained all homogeneous components of y_i, then it would contain their sum y_i, which it doesn't. So, some homogeneous component of y_i is not contained in $\mathfrak{p}_r$, but $\mathfrak{p}_i$ is homogeneous, so it does contain this component. Thus, without loss of generality, we can assume each y_i is homogeneous by replacing it with this homogeneous component if necessary. Similarly, we can choose $z \in I \setminus \mathfrak{p}_r$ homogeneous. But then the product $a = z \cdot y_1 \cdots y_{r-1}$ is homogeneous and not in $\mathfrak{p}_r$ by primality.

By the induction hypothesis, we can choose such a b. If $b \notin \bigcup_i \mathfrak{p}_i$ then we're done, so assume otherwise. Since $b \notin \mathfrak{p}_i$ for $i < r$, we must have $b \in \mathfrak{p}_r$. Then $a \notin \mathfrak{p}_r$, so $a^{\deg b} \notin \mathfrak{p}_r$ by primality, and so $a^{\deg b} + b^{\deg a} \notin \mathfrak{p}_r$. For $i < r$, we have $a \in \mathfrak{p}_i$ so $a^{\deg b} \in \mathfrak{p}_i$, but $b \notin \mathfrak{p}_i$, so $b^{\deg a} \notin \mathfrak{p}_i$ by primality. Then we again get $a^{\deg b} + b^{\deg a} \notin \mathfrak{p}_i$. So we have found a homogeneous element (of degree $(\deg a)(\deg b)$) of I not contained in any $\mathfrak{p}_i$ as desired. □

Exercise (2.5.6).

Proof. Let $\mathfrak{p}$ be a prime ideal of A containing I, and so in particular containing x. Suppose we have a chain of primes $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$. By a lemma in the section, there is another chain of primes $\mathfrak{q}_1 \subsetneq \cdots \mathfrak{q}_n = \mathfrak{p}$ with $x \in \mathfrak{q}_1$. This gives a chain of prime ideals in A/xA ending with $\mathfrak{p}/xA$, and so we conclude $n - 1 \leq \operatorname{ht}(\mathfrak{p}/xA)$, i.e. $n \leq \operatorname{ht}(\mathfrak{p}/xA) + 1$. Taking the supremum over all n, i.e. over all chains of primes, we conclude that $\operatorname{ht}(\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p}/xA) + 1$. Taking the minimum over all $\mathfrak{p}$ containing I gives us the conclusion we want: $\operatorname{ht}(I) \leq \operatorname{ht}(I/xA) + 1$. In particular, we're using here the fact that as $\mathfrak{p}$ varies over all primes of A containing I, $\mathfrak{p}/xA$ enumerates all primes of A/xA containing I/xA, since the ideals of A/xA correspond to ideals of A containing x, primes are preserved under this correspondence, and any prime containing I automatically contains x.

On the other hand, the inequality $\operatorname{ht}(I/xA) \leq \operatorname{ht}(I)$ is obvious, since any chain of primes in A/xA lifts to one in A. Thus, if the previous inequality is strict, we must have $\operatorname{ht}(I/xA) = \operatorname{ht}(I)$. Choose a prime $\mathfrak{p}$ containing I with $\operatorname{ht}(I/xA) = \operatorname{ht}(\mathfrak{p}/xA)$ and a chain of primes in A/xA ending with $\mathfrak{p}/xA$. This lifts to a chain of primes in A containing x, and the first prime in this chain must be minimal in A, else we would get $\operatorname{ht}(I) > \operatorname{ht}(I/xA)$. So, this first prime is a minimal prime of A containing x.

Suppose now that I is contained in the union of minimal primes of A. Since A is noetherian, there are only finitely many of these, and by prime avoidance, it must then be contained in one of them. But then $\operatorname{ht}(I) = 0$. Conversely, suppose I is not contained in the union of minimal primes. Then choose $x \in I$ that is in no minimal prime and note that by the above argument, $\operatorname{ht}(I) = \operatorname{ht}(I/xA) + 1 \geq 1$.

Finally, we show the last claim by induction on r. If $r = 0$, then the statement is trivial by taking $J = 0$. Suppose now $r \geq 1$. By the above, $\operatorname{ht}(I) = r \geq 1$ means that we can choose $x \in I$ not contained in any minimal prime of A. Then $\operatorname{ht}(I/xA) = \operatorname{ht}(I) - 1 = r - 1$, so by induction, we can choose an ideal J' generated by $r-1$ elements with $\operatorname{ht}(J') = r - 1$ and $\operatorname{ht}((I/xA)/J') = 0$. Lifting these generators gives an ideal J of A with $J/xA = J'$. Then $J + xA$ is generated by r elements and since x is still not contained in any minimal prime of A, we conclude that $\operatorname{ht}(J + xA) = \operatorname{ht}((J + xA)/xA) + 1 = \operatorname{ht}(J') + 1 = r$. Finally, note that since $(I/xA)/J'$ has height zero, there is a minimal prime of $(A/xA)/J' \cong A/(J + xA)$ containing I. But this isomorphism shows that there is a minimal prime of $A/(J + xA)$ containing I, and so $\operatorname{ht}(I/(J + xA)) = 0$. This completes the proof using the ideal $J + xA$. □

Exercise (2.5.7).

Proof. Note first that dimension is a topological property, so we may assume that all schemes are reduced. Indeed, replacing a scheme with its reduced scheme structure does not change the topology at all, and hence preserves the dimension.

Note also that it suffices to show the claim when Y is an affine variety. Indeed, we can cover Y by affine varieties, and the preimage of this gives an open cover of X by algebraic varieties. Showing the claim on each of these restricted maps then gives the result overall since the dimension of Y can be computed locally on this cover and the dimension of X can be computed locally on the preimage cover. So, write $Y = \operatorname{Spec} A$ for A a finitely generated k-algebra.

Let S denote the set of generic points of irreducible components of X. Since X is noetherian, this is a finite set, and purely topologically, we have:

$$\bigcup_{x \in S} V(f(x)) = \bigcup_{x \in S} \overline{f(x)} = \overline{f(S)} = \overline{f(\overline{S})} = \overline{f(X)} = Y$$

So, in particular, any minimal prime of A is the image of some generic point of X. Choose such a minimal prime, corresponding to a generic point $\xi_Y \in Y$ and a preimage $\xi_X \in X$ also generic. On these irreducible components, f is dominant, so we have an injective map of sheafs, which induces an injection $\mathscr{O}_{Y,\xi_Y} \to \mathscr{O}_{X,\xi_X}$. But (after passing to the reduced subscheme structure if necessary) this means that the transcendence degree of the latter is at least as much as the transcendence degree of the former over k. So, the dimension of this irreducible component of Y is at most the dimension of this irreducible component of X, which is itself at most the dimension of X. Taking the supremum over all $\xi_Y \in Y$ gives the result. ☐

Exercise (2.5.8).

Proof. Note that $\operatorname{Spec} \mathscr{O}_K = \{(0), (t)\}$ and that $\operatorname{Spec} A = \{(0) \times k, K \times (0)\}$. Then, in fact, f_φ is a bijection since $\phi^{-1}((0) \times k) = (0)$ and $\phi^{-1}(K \times (0)) = (t)$. However, the two primes in $\mathscr{O}_K$ form a chain, while the two primes of A are incomparable, so $\dim \mathscr{O}_K = 1$ and $\dim A = 0$. Finally, A is a finitely generated $\mathscr{O}_K$-algebra since both $K = \mathscr{O}_K[x]/(tx - 1)$ and $k = \mathscr{O}_K/(t)$ are. ☐

Exercise (2.5.9).

Proof. Let $x \in X(k)$. To show $\{x\}$ is closed, it suffices to show it is closed in any affine chart $\operatorname{Spec} A$ containing x. Suppose instead that it is not, so that for some A, x is a non-maximal prime. Then $k = k(x) = \operatorname{Frac}(A/x)$, but A/x is a domain that is not a field, so it is at least 1-dimensional, making $\operatorname{Frac}(A/x)$ have transcendence degree at least 1 over k. This is a contradiction, so we must have that $\{x\}$ is closed. More generally, we've shown that if $k(x)$ is algebraic over k, then x is closed in X.

For X an algebraic variety, we have the converse. Indeed, if $x \in X$ is closed, then picking an affine chart, we get that x is a maximal ideal of a finitely generated k-algebra A. But then $k(x) = A/x$ is a field extension of k that is finitely generated as a k-algebra, and so is a finite extension by Zariski's lemma. In particular, $k(x)$ is algebraic over k. ☐

Exercise (2.5.10).

Proof. For $X = \mathbb{A}^1_k$, the result is immediate: each Y_n is closed, so it is of the form $V(I)$ for I an ideal of $k[T]$. But this is a PID, so $I = (f)$. If $f = 0$, then $V(I) = X$, which doesn't have strictly smaller dimension than X, so we must have $f \neq 0$. But then f only has finitely many prime factors, so $V(I)$ is finite, i.e. each Y_n is finite. So, $\bigcup_n Y_n$ is a countable set, while X is uncountable, since each $(T - a)$ is a maximal ideal of $k[T]$ for each $a \in k$ and there are uncountably many of these.

This now implies the case for $X = \mathbb{A}^m_k$ by induction on m. Indeed, first again write Y_n as $V(I)$ for some ideal I. Since $k[T_1, \ldots, T_m]$ is Noetherian, $I = (f_1, \ldots, f_r)$ is finitely generated. Considering each f_i as a polynomial in T_m with coefficients in $k[T_1, \ldots, T_{m-1}]$, they each have finitely many roots. In other words, overall, I is contained in only finitely many ideals of the form $(T_m - a)$ for $a \in k$. So, the set $S = \{(Y_n, a) \mid (T_m - a) \in Y_n\}$ is countable, as a countable union of finite sets. On the other hand, for $a \in k$, consider the map $\varphi_a : k[T_1, \ldots, T_m] \to k[T_1, \ldots, T_{m-1}]$ that maps T_m to a and is the identity on the rest. This induces a map $f_a : \mathbb{A}^{m-1}_k \to \mathbb{A}^m_k$ and if $\bigcup_n Y_n = \mathbb{A}^m_k$, then $\bigcup_n f_a^{-1}(Y_n) = \mathbb{A}^{m-1}_k$. By induction, this implies $f_a^{-1}(Y_n) = \mathbb{A}^{m-1}_k$ for some n. But then $(T_m - a) = \ker(\varphi_a) = \varphi^{-1}(0) = f_a(0) \in Y_n$. This shows that the set S above is uncountable, since for each $a \in k$ it contains (Y_n, a) for some n.

Now, if $X = \operatorname{Spec} B$ is an affine variety, so B is a finitely generated k-algebra, then we can write B as a finite $A = k[T_1, \ldots, T_m]$-module for a subring $A \subseteq B$. But then the induced map $f : X \to \operatorname{Spec} A$ is closed. Indeed, if $V(I)$ is a closed subset of X, then $f(V(I)) = V(A \cap I)$. One containment is obvious: if $Q \subseteq B$ is a prime containing I, then $Q \cap A$ is a prime of A containing $A \cap I$. For the other containment, note that if $P \in \operatorname{Spec} A$ contains $A \cap I$, then the composition $A \to B \to B/I$ is integral, so satisfies lying-over, giving a prime Q of B containing I with $Q \cap A = P$.

But now, if $X = \bigcup_n Y_n$, then $\operatorname{Spec} A = \bigcup_n f(Y_n)$ (again using lying-over for surjectivity of f), and so by the previous work $f(Y_n) = \operatorname{Spec} A$ for some n. But $\dim(f(Y_n)) \leq \dim(Y_n) < \dim(X) = \dim(\operatorname{Spec} A)$ (first inequality from a prior exercise) so this cannot be.

Finally, if X is now an arbitrary algebraic variety and $X = \bigcup_n Y_n$, then $U = \bigcup_n (U \cap Y_n)$ for any affine $U \subseteq X$. But $\dim(U \cap Y_n) \leq \dim(Y_n) < \dim(X) = \dim(U)$, so this cannot be either. ☐

Exercise (2.5.11).

Proof. This is just the sheaf condition.

Let $U = \operatorname{Spec} A$ be an affine chart in X. Since $\dim(X) = 0$, $\dim(U) = 0$, so every prime of A is maximal, i.e. every point of U is closed. Since there are only finitely many of them, this means every subset is closed, so every point in U is open in U, whence it is open in X as well. So X has the discrete topology.

But for rings A and B, note that the coproduct of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ in the category of schemes is $\operatorname{Spec}(A \times B)$. In particular, the finite disjoint union of affine schemes is again affine. But each singleton $\{x\}$ is affine; indeed, the affine sets form a base for the topology on X and $\{x\}$ is open, so it contains an open affine U that contains x, which must be $U = \{x\}$ itself. So, now, X is affine as the disjoint union of the finitely many singletons.

Note that we've already shown that X has the discrete topology. Hence if $X = \operatorname{Spec} A$, from the previous two points, we get

$$A = \mathscr{O}_X(X) = \prod_{x \in X} \mathscr{O}_X(\{x\}) = \prod_{x \in X} \mathscr{O}_{X,x} = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}}$$

as claimed.

For a counterexample when A has positive dimension, consider A to be a DVR with uniformizer t and field of fractions K. Then $\dim A = 1$ and $\operatorname{Spec} A = \{(0), (t)\}$. The singleton $\{(t)\}$ is not open since $\{(0)\}$ is not closed, as (0) is not maximal. So not every singleton is open, unlike above. Further, we have $\bigoplus_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}} = K \oplus A$. But this is not isomorphic to A since $\operatorname{Spec}(K \oplus A) = \{(0) \oplus A, K \oplus (0), K \oplus (t)\}$ has 3 points, rather than 2. ☐

Exercise (2.5.12).

Proof. More generally, if A is a ring and $f \in A$, then $D(f) \subseteq \operatorname{Spec}(A)$ is isomorphic to $\operatorname{Spec}(A_f)$, and so a principal open subset of an affine scheme is affine. Similarly, if B is graded and $f \in B_+$ is homogeneous, then $D_+(f) \cong \operatorname{Spec}(B_{(f)})$ and so is also affine.

By the sheaf condition, $\mathscr{O}_{\mathbb{A}^n_k}(X)$ is the kernel of the map:

$$\prod_i \mathscr{O}_{\mathbb{A}^n_k}(D(f_i)) \to \prod_{i,j} \mathscr{O}_{\mathbb{A}^n_k}(D(f_i) \cap D(f_j))$$

On the one hand, $\mathscr{O}_{\mathbb{A}^n_k}(D(f_i)) = k[T_1, \dots, T_n]_{f_i}$. Then,

$$D(f_i) \cap D(f_j) = \mathbb{A}^n_k \setminus V((f_i) \cap (f_j)) = D(\operatorname{lcm}(f_i, f_j))$$

So $\mathscr{O}_{\mathbb{A}^n_k}(D(f_i) \cap D(f_j)) = k[T_1, \dots, T_n]_{f_{ij}}$ where $f_{ij} = \operatorname{lcm}(f_i, f_j)$. Composing with an injection doesn't change the kernel, so to simplify, we can embed everything in $K = k(x_1, \dots, x_n)$, the field of fractions. Then, an element in the first ring looks like a tuple $(g_i f_i^{-t_i})_i$, and if it is in the kernel, then $g_i f_i^{-t_i} = g_j f_j^{-t_j}$ in K for every i, j. Fix some index i and by factorization in $k[x_1, \dots, x_n]$, write $g_i f_i^{-t_i} = a/b$ for a, b coprime; I claim that $b \mid f^r$ for some $r \in \mathbb{Z}$. Indeed, it suffices to show that each prime factor of b also divides f. Let $p \mid b$ be a prime factor, and write ν_p for the p-adic valuation. For contradiction, suppose $\nu_p(f) = 0$. Then

$$t_i \nu_p(f_i) = \nu_p(a f_i^{t_i}) - \nu_p(a) = \nu_p(b g_i) = \nu_p(b) + \nu_p(g_i)$$

where we've used the fact that $p \nmid a$ since a, b are coprime. On the other hand, p doesn't divide the gcd of a set of polynomials, so it fails to divide one of them, say $p \nmid f_j$. By the kernel condition,

$$t_i \nu_p(f_i) = \nu_p(f_i^{t_i} g_j) = \nu_p(f_j^{t_j} g_i) = \nu_p(g_i)$$

Equating these gives $\nu_p(b) = 0$, contrary to assumption. So, indeed, $p \mid f$ and so $b \mid f^r$ for some r.

But now, what we've shown is that any element of the kernel is a tuple where each element can be written in the form $a_i f^{-r_i}$. But the kernel condition also implied that each of these elements are equal (in K), and so really they are the image of a single element $a f^{-r} \in k[x_1, \dots, x_n]_f$. Conversely, it is clear that the image of any such element gives a tuple in the kernel. So, $\mathscr{O}_{\mathbb{A}^n_k}(X) = k[x_1, \dots, x_n]_f$ as claimed.

Note that

$$X = \bigcup_i D(f_i) = \mathbb{A}^n_k \setminus \bigcap_i V(f_i) = \mathbb{A}^n_k \setminus V\left(\sum_i (f_i)\right) \subseteq \mathbb{A}^n_k \setminus V(f) = D(f)$$

since $f \mid f_i$ for each i. If X is affine, then this containment exhibits X as a localization of $D(f)$ which is isomorphic to it, whence we get $X = D(f)$ is principal. [There's something wrong here; come back to it]

Let Z be an irreducible closed subset of $\mathbb{P}^n_k$ of dimension $n-1$. In each coordinate chart, this is an irreducible closed subset of $\mathbb{A}^n_k$ of dimension $n-1$, so it equals $V(P)$ for a height 1 prime P, which is principal since $k[x_1, \ldots, x_n]$ is a UFD. So, Z is principal in each chart and glues to be principal overall. [Fill in the details: why does Z not drop in dimension in charts, why can we glue, etc.]

[Not sure]. □

Exercise (2.5.13).

Proof. First, suppose that X is an integral algebraic variety over k of dimension n. We've already seen that this implies that the transcendence degree of $K(X)$ is then $\dim(X) = n$ as desired. It is also finitely generated as a field extension over k. Indeed, choose an affine subvariety $U = \operatorname{Spec} A$, where A is a domain and a finitely generated k-algebra, and note that $K(X) = \mathscr{O}_{X,\xi} \cong \operatorname{Frac}(\mathscr{O}_X(U)) = \operatorname{Frac}(A)$, which is generated (as a field over k) by the same generators of A over k.

Conversely, suppose K has transcendence degree n over k and is finitely generated. Choose a transcendence basis $\{f_1, \ldots, f_n\} \subseteq K$, so that K is algebraic over $L = k(f_1, \ldots, f_n)$; let $A = k[f_1, \ldots, f_n]$. In fact, K is also finitely generated over L since it is finitely generated over k, so we can choose generators $g_1, \ldots, g_r$ giving $K = L(g_1, \ldots, g_r)$. Let $L_i = L(g_1, \ldots, g_i)$ and $A_i = A[g_1, \ldots, g_i]$, so we get a tower of extensions $L \subseteq L_1 \subseteq \cdots \subseteq L_r = K$. At each step, we adjoin a single element g_i with minimal polynomial $m_i \in L_{i-1}[T]$. In fact, by clearing denominators, we can find $a \in A$ such that the minimal polynomial of $ag_i \in A_{i-1}[T]$ instead. Replacing g_i with ag_i doesn't change any of the fields involved, so without loss of generality, we may assume $m_i \in A_{i-1}[T]$ for each i.

Now, let $X = \operatorname{Spec} A_r$. We've constructed the tower of rings to be integral extensions at each step, so $\dim(X) = \dim(A_r) = \dim(A) = n$. It is also clear that A_r is a finitely generated k-algebra and a domain, so X is an integral algebraic variety over k. Finally, $K(X) = \operatorname{Frac}(A_r) = L_r = K$ as desired. So, we're done.

[I'm not totally clear on how to make X projective. Maybe it already is? I should try to embed it in $\mathbb{P}^{n+r}_k$ or something. Something something ample divisor.] □

Exercise (2.5.14).

Proof. For $x \in L$, let $m_x : L \to L$ be the multiplication by x map. Picking a basis for L over K allows us to write m_x as a matrix with coefficients in K, whence $N(x) = \det(m_x) \in K$. Multiplicativity is also clear, since for $x, y \in L$: $N(xy) = \det(m_{xy}) = \det(m_x \circ m_y) = \det(m_x)\det(m_y) = N(x)N(y)$.

Let $b \in B$, and let

$$f(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$$

be the minimal polynomial of b over K. Consider the tower $K \subseteq K(b) \subseteq L$. Fixing a basis $c_1, \ldots, c_r$ of L over $K(b)$ gives the basis $\{c_i b^j \mid 1 \le i \le r, 0 \le j \le n-1\}$ of L over K. With respect to this basis, multiplication by b has a simple form: it is block diagonal, where each block is the matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

□

Exercise (2.5.15).

Proof. □