Qing Liu, Algebraic Geometry and Arithmetic Curves: Solutions **Name:** Abhay Goel
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Exercise (3.1.1).

Proof. Define $(f,g) : Z \to X \times_S Y$ by $(f,g)(z) = (f(z), g(z))$. To see that this is actually an element of $X \times_S Y$ (and not just $X \times Y$), note that $\phi(f(z)) = (\phi \circ f)(z) = (\psi \circ g)(z) = \psi(g(z))$ as needed. Further, this map satisfies the desired condition since:

$$(p \circ (f,g))(z) = p(f(z), g(z)) = f(z) \text{ and } (q \circ (f,g))(z) = q(f(z), g(z)) = g(z)$$

for all $z \in Z$. The uniqueness of (f,g) is obvious. □

Exercise (3.1.2).

Proof. Suppose first that f is an isomorphism with inverse $f^{-1} : Y \to X$. Then we have $(f^{-1}(T) \circ f(T))(g) = f^{-1} \circ (f \circ g) = g$ and $(f(T) \circ f^{-1}(T))(g) = f \circ (f^{-1} \circ g) = g$, so $f(T)$ is a bijection (with inverse $f^{-1}(T)$).

Now, suppose that $f(T)$ is a bijection for every T. In particular, for $T = Y$, we have $\mathrm{id}_Y \in Y(T)$, so by surjectivity there is some $g : Y \to X$ such that $f \circ g = f(T)(g) = \mathrm{id}_Y$. Then, taking $T = X$ instead, we have:

$$f(T)(g \circ f) = f \circ (g \circ f) = (f \circ g) \circ f = \mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X = f(T)(\mathrm{id}_X)$$

so by injectivity $g \circ f = \mathrm{id}_X$. But we've shown that g is a two-sided inverse to f, so f is an isomorphism. □

Exercise (3.1.3).

Proof. Let ξ_X, ξ_Y denote the generic points of X, Y, respectively. We have:

$$\overline{\{f(\xi_X)\}} = \overline{f(\overline{\{\xi_X\}})} = \overline{f(X)} = Y$$

where the last equality comes from f being dominant. So $\{f(\xi_X)\}$ is a point whose closure is the full space Y. In other words, $f(\xi_X) = \xi_Y$. So, the generic fiber $X_{\xi_Y} = f^{-1}(\xi_Y)$ contains ξ_X, and so is a dense subset of X. □

Exercise (3.1.4).

Proof. Note that we have the map $f : X \to Y$ and the inclusion $i : V \to Y$. Construct the fiber product $X \times_Y V$ of these maps, and let $p : X \times_Y V \to X$ be the first projection. I claim that p induces an isomorphism onto its image, $f^{-1}(V)$. Showing this would complete the proof, since then $f^{-1}(V) \cong X \times_Y V$, which is the fiber product of affine schemes over an affine scheme, and is thus affine (given by the tensor product).

Let $U = f^{-1}(V)$. First, the fact that the image of p lies in U is obvious. Indeed, for $z \in X \times_Y V$, $f(p(z)) = i(q(z)) \in i(V) = V$, where q is the other projection. So $p(z) \in U$.

We'd like to construct the reverse map. For this, note that we have the inclusion map $\iota : U \to X$ and the restriction $f|_U : U \to V$, and clearly $i \circ (f|_U) = f \circ \iota$. So, by the universal property of the fiber product, we get a map $g : U \to X \times_Y V$ such that $\iota = p \circ g$ and $f|_U = q \circ g$. Note that this first fact gives that $p \circ g$ is the identity on U. For the other direction, note that $\mathrm{id}_{X \times_Y V}$ is the unique map h from $X \times_Y V$ to itself satisfying $p = p \circ h$ and $q = q \circ h$. But

$$p \circ (g \circ p) = (p \circ g) \circ p = p$$

and

$$q \circ (g \circ p) = (q \circ g) \circ p = f|_U \circ p = i|_V \circ q = \mathrm{id}_V \circ q = q$$

So, $g \circ p = \mathrm{id}_{X \times_Y V}$ and we've demonstrated the required isomorphism. □

Exercise (3.1.5).

Proof. Let $U \subseteq X$ and $V \subseteq Y$ be affine open neighborhoods. Note that we have a commutative diagram:

$$\begin{array}{ccccc}
U \times_k V & \xrightarrow{*} & U \times_k Y & \longrightarrow & U \\
\downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *} \\
X \times_k V & \xrightarrow{*} & X \times_k Y & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
V & \xrightarrow{*} & Y & \longrightarrow & \operatorname{Spec} k
\end{array}$$

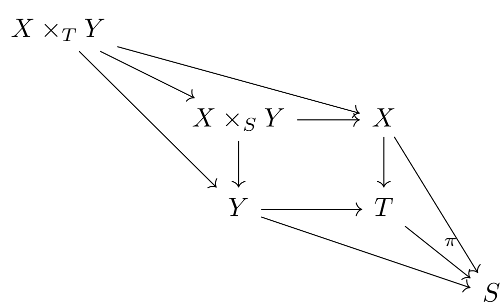
In fact, each of the smaller subsquares is a fiber product by repeated application of the fact that $(A \times_S B) \times_B C \cong A \times_S C$ with the appropriate morphisms. Further, note that the morphisms $U \hookrightarrow X$ and $V \hookrightarrow Y$ are open immersions and since this is preserved by base change, each morphism above labelled "$*$" is an open immersion. Finally, the composition of open immersions is an open immersion, so we get that $U \times_k V$ is an open subset of $X \times_k Y$. But the former is the fiber product of affine schemes over an affine base scheme, so is itself affine. In fact, it is the tensor product of two finitely generated k-algebras, so is itself a finitely generated k-algebra. Doing this for all U in a cover of X and all V in a cover of Y shows that $X \times_k Y$ is itself an algebraic variety, and so $\dim(X \times_k Y) = \dim(U \times_k V)$, $\dim X = \dim U$, and $\dim Y = \dim V$.

So, we have reduced to the affine case. Namely, suppose A, B are finitely generated k-algebras. Then we can choose noether normalizations so that $k[x_1, \ldots, x_n] \subseteq A$ and $k[y_1, \ldots, y_m] \subseteq B$ are finite extensions. Tensoring over k gives that $A \otimes_k B \supseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is a finite extension, and so $\dim(A \otimes_k B) = n + m = \dim(A) + \dim(B)$, completing the proof. □

Exercise (3.1.6).

Proof. First consider the case that π is an immersion (open or closed). Then π is injective on points and $\pi^\#$ is surjective. So, on points, the fact that $\pi \circ f = \pi \circ g$ implies $f = g$, and on structure sheaves, the fact that $f^\# \circ \pi^\# = g^\# \circ \pi^\#$ implies that $f^\# = g^\#$. Hence $f = g$ as maps of sheaves as claimed. When π is induced by a localization, then it is still injective on points, but is not surjective on the level of rings. But $f^\# : F^{-1}A \to B$ is fully determined by the map $f^\# \circ \pi^\# : A \to B$ since $f^\#(a/u) = f^\#(\pi^\#(a))/f^\#(\pi^\#(u))$. On principal open sets, this is exactly the condition we have, so we conclude that $f = g$.

This statement holds in any category with fiber products given that π is a monomorphism. First, note that since the compositions $X \times_T Y \to X \to T$ and $X \times_T Y \to Y \to T$ are equal, composing with the map $\pi : T \to S$ retains that equality. Hence, we get the claimed map $X \times_T Y \to X \times_S Y$.



For the reverse map, note that the maps $X \times_S Y \to X \to T \xrightarrow{\pi} S$ and $X \times_S Y \to X \to T \xrightarrow{\pi} S$ are equal. But by the above, π is mono, so we get that $X \times_S Y \to X \to T$ and $X \times_S Y \to Y \to T$ are equal. So, we get a map $X \times_S Y \to X \times_T Y$. It is immediate from a diagram chase that these two maps are pairwise inverse, so that the fiber products are canonically isomorphic.

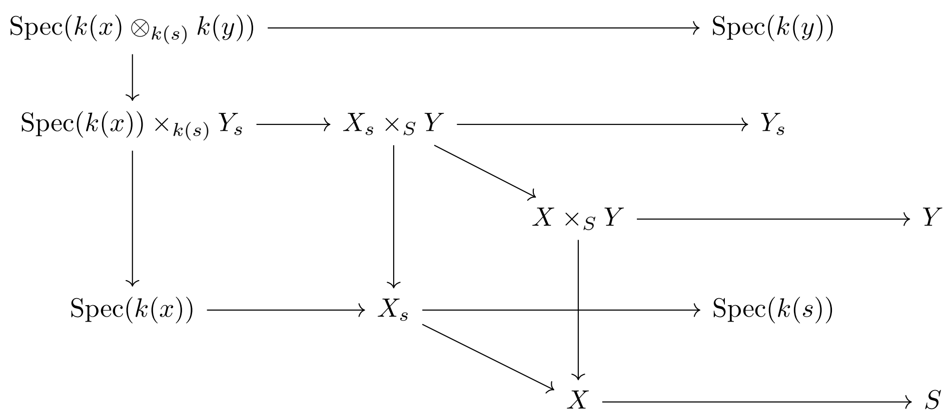
Finally, take the case $X = Y = T = \operatorname{Spec} \mathbb{Q}(i)$ and $S = \operatorname{Spec} \mathbb{Q}$ with $\pi : T \to S$ induced from the inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}(i)$. Then $X \times_T Y = T \times_T T = T = \operatorname{Spec} \mathbb{Q}(i)$, but $X \times_S Y = T \times_S T = \operatorname{Spec}(\mathbb{Q}(i) \times_{\mathbb{Q}} \mathbb{Q}(i)) = \operatorname{Spec}(\mathbb{Q}(i) \oplus \mathbb{Q}(i))$. These cannot be isomorphic since the former is a single point while the latter is a two-point space. □

Exercise (3.1.7).

Proof. Fix $s \in S$, $x \in X_s$, and $y \in Y_s$ as noted. The set $\{z \in X \times_S Y : p(z) = x, q(z) = y\}$ can be identified with the fiber of a fiber: $((X \times_S Y)_x)_y$. This allows us to compute it directly:

$$\begin{aligned}
((X \times_S Y)_x)_y &= (\operatorname{Spec}(k(x)) \times_X (X \times_S Y))_y \\
&= (\operatorname{Spec}(k(x)) \times_{X_s} (X_s \times_X (X \times_S Y)))_y \\
&= (\operatorname{Spec}(k(x)) \times_{X_s} (X_s \times_S Y))_y \\
&= (\operatorname{Spec}(k(x)) \times_{X_s} (X_s \times_{k(s)} (\operatorname{Spec}(k(s)) \times_S Y)))_y \\
&= (\operatorname{Spec}(k(x)) \times_{X_s} (X_s \times_{k(s)} Y_s))_y \\
&= (\operatorname{Spec}(k(x)) \times_{k(s)} Y_s)_y \\
&= (\operatorname{Spec}(k(x)) \times_{k(s)} Y_s) \times_{Y_s} \operatorname{Spec}(k(y)) \\
&= \operatorname{Spec}(k(x)) \times_{k(s)} \operatorname{Spec}(k(y)) \\
&= \operatorname{Spec}(k(x) \otimes_{k(s)} k(y))
\end{aligned}$$

which completes the claim. To visualize the above argument, consider the diagram:



∎

Exercise (3.1.8).

Proof. Let $f : X \to S$ be surjective and $g : Y \to S$ be any morphism. We wish to show that $q : X \times_S Y \to Y$ surjects, so let $y \in Y$. Then, let $s = g(y)$, and since f is surjective, pick any x such that $f(x) = s$. Then, by the previous exercise, we have a homeomorphism

$$\operatorname{Spec}(k(x) \otimes_{k(s)} k(y)) \to \{z \in X \times_S Y : p(z) = x, q(z) = y\}$$

In particular, the former is nonempty, so the latter is nonempty, giving $z \in X \times_S Y$ such that $q(z) = y$ as desired. ∎

Exercise (3.1.9).

Proof. More generally, note that if A, B are R-algebras with multiplicative subsets S, T, respectively, then $S^{-1}A \otimes_R T^{-1}B \cong (S \otimes T)^{-1}(A \otimes_R B)$, where $S \otimes T = \{s \otimes t : s \in S, t \in T\}$. In our case, this gives that $k(u) \otimes_k k(v)$ is the localization of $k[u] \otimes_k k[v] \cong k[u, v]$ at $T' = T \otimes T$, where T is the nonzero elements of $k[u], k[v]$, respectively. Under the above isomorphism, then, we have that a general element $P \otimes Q \in T'$ corresponds to the polynomial $P(u)Q(v) \in k[u, v]$, with both P, Q nonzero, exactly as claimed.

Let $\mathfrak{m} \in k[u, v]$ be maximal, and suppose for contradiction that there is no nonzero polynomial $P(u) \in \mathfrak{m}$. In particular, this shows that $\mathfrak{m}$ is disjoint from $T = \{P(u) \in k[u, v] : P \neq 0\}$, which is a multiplicative subset of $k[u, v]$. So, $\mathfrak{m}$ corresponds to a maximal ideal $\mathfrak{m}'$ of $T^{-1}k[u, v] = k(u)[v]$. So, $k(u)[v]/\mathfrak{m}'$ is a field extension of $k(u)$, and hence is transcendental over k. On the other hand, $k[u, v]/\mathfrak{m}$ is a field extension of k, and it is finitely generated as a k-algebra, so it is in fact a finite extension (Zariski Lemma). Localization commutes with taking quotients, so we would then get that $T^{-1}(k[u, v]/\mathfrak{m})$ is the localization of a field at a multiplicative subset, which must therefore be just $k[u, v]/\mathfrak{m}$ itself, and so we have our contradiction as this is supposed to be both a finite extension of k and transcendental over k. As noted, we thus conclude that $T' \cap \mathfrak{m} \neq \emptyset$.

Now, suppose that $\mathfrak{m}$ is maximal in $A = T'^{-1}k[u, v]$. Then $\mathfrak{m}$ corresponds to some prime ideal $\mathfrak{p}$ of $k[u, v]$ that is disjoint from T'. Let's consider the different cases for $\operatorname{ht}(\mathfrak{p})$. Since $\dim(k[u, v]) = 2$, we must have $\operatorname{ht}(\mathfrak{p}) \leq 2$. If $\operatorname{ht}(\mathfrak{p}) = 2$, then $\mathfrak{p}$ is maximal, but then we've just shown that $\mathfrak{p}$ is not disjoint from T'. If $\operatorname{ht}(\mathfrak{p}) = 0$, then $\mathfrak{p} = (0)$ itself, in which case it corresponds to $\mathfrak{m} = 0A$. This is a contradiction since (0) is not maximal. So, we conclude that $\operatorname{ht}(\mathfrak{p}) = 1$, in which case $\mathfrak{p} = gk[u, v]$ since $k[u, v]$ is a UFD. In order for this to be prime, we must have that g is irreducible, and in order for it to be disjoint from T', we must at least have that $g \notin k[u] \cup k[v]$. Then, it is clear that $\mathfrak{m} = gA$ is of the desired form.

Conversely, suppose that $g \in k[u, v] \setminus (k[u] \cup k[v])$ is irreducible. Suppose, for contradiction, that $T' \cap gk[u, v] \neq \emptyset$. In other words, we may write $P(u)Q(v) = g(u, v)h(u, v)$ for some P, Q, h. But now, g is irreducible and $k[u, v]$ is a UFD, so g must be a factor of PQ, which means (WLOG) it is a factor of P. But $k[u]$ is also a UFD, so we can factorize P in $k[u]$, showing that each of its irreducible factors is in $k[u]$ as well, giving $g \in k[u]$, contrary to assumption. So, the two sets are indeed disjoint, which gives that gA is prime in A. But by again considering the height, it must actually be maximal: if $gA \subsetneq \mathfrak{m}$ in A, then this corresponds to $gk[u, v] \subsetneq \mathfrak{m}'$ in $k[u, v]$ disjoint from T', but then $\mathfrak{m}'$ is a height 2 prime of $k[u, v]$, which is thus maximal and cannot be disjoint from T'. So we're done.

We've already seen repeatedly that $\dim(A) = 1$. To see that $\operatorname{Spec}(A)$ is infinite, note that for each $n \geq 1$, $g_n(u, v) = u - v^n$ is irreducible and not in $k[u] \cup k[v]$. So, each $g_n A$ is maximal in A. It only remains to show they are distinct, but since this is true in $k[u, v]$, it is true in A as well. ∎

Exercise (3.1.10).

Proof. First, note that the scheme morphisms $X \times_S Y \to X$ and $X \times_S Y \to Y$ include as part of their data continuous maps $sp(X \times_S Y) \to sp(X)$ and $sp(X \times_S Y) \to sp(Y)$, respectively, which commute after projecting down to $sp(S)$. Hence by the universal property of the fiber product in the category of topological spaces, we get a continuous map $f : sp(X \times_S Y) \to sp(X) \times_{sp(S)} sp(Y)$ as claimed.

Explicitly, we can identify a point of $sp(X) \times_{sp(S)} sp(Y)$ as a pair (x, y) with $x \in X$ and $y \in Y$ such that $\pi(x) = \rho(y) \in S$. But then we've seen (3.1.7) that there is some $z \in X \times_S Y$ that projects to both x and y under the scheme morphisms. By the construction of f, we get $f(z) = (x, y)$, and so f is indeed surjective.

Indeed,

$$X \times_S Y = \operatorname{Spec}(\mathbb{C} \times_{\mathbb{R}} \mathbb{C}) = \operatorname{Spec}(\mathbb{C}[x]/(x^2 + 1)) \cong \operatorname{Spec}(\mathbb{C} \oplus \mathbb{C})$$

as claimed. But $sp(X)$, $sp(Y)$, and $sp(S)$ are all singletons, so the topological fiber product is also. Hence f cannot inject since the domain has two points.

Again, in this case each space is a singleton, so the only fiber is all of $\operatorname{Spec}(A)$. But we've already shown that this is an infinite set.

Note that in this case, $X \times_S Y = \mathbb{A}^2_k = \operatorname{Spec}(k[x, y])$, and $sp(X) \times_{sp(S)} sp(Y) = sp(X) \times sp(Y) = \operatorname{Spec}(k[x]) \times \operatorname{Spec}(k[y])$ since $sp(S)$ is a singleton. We can also describe f explicitly: for a prime ideal $P \subseteq k[x, y]$, we have $f(P) = (P \cap k[x], P \cap k[y])$.

Suppose first that k is infinite. Consider the open subset $D(x - y) \subseteq \mathbb{A}^2_k$, and suppose that $f(D(x - y))$ is open. Then, it contains an open rectangle since they form a basis for the product topology, say $U \times V$ for $U \subseteq \operatorname{Spec}(k[x])$ and $V \subseteq \operatorname{Spec}(k[y])$. Identifying the two copies of $\mathbb{A}^1_k$ as $\operatorname{Spec}(k[t])$, we get that $U, V \subseteq \mathbb{A}^1_k$. Since $\mathbb{A}^1_k$ is irreducible, we must have that $U \cap V$ is a nonempty open set. So, its complement is a proper closed set $V(I)$, and since $k[t]$ is a PID, we may assume $I = (g)$ is principal. But then g has finitely many roots in k, so there is some $a \in k$ with $g(a) \neq 0$. Thus, $(g) \not\subseteq (t - a)$. In other words, $(t-a) \notin \mathbb{A}^1_k \setminus (U \cap V)$, so $(t-a) \in U \cap V$. Reverting to our original notation, this means that $((x-a), (y-a)) \in f(D(x-y))$, so there is some prime P with $P \cap k[x] = (x - a)$ and $P \cap k[y] = (y - a)$. Thus, $P \supseteq (x - a, y - a)$, but this ideal is maximal, so $P = (x - a, y - a)$. However, then $x - y = (x - a) - (y - a) \in P$, and so $P \notin D(x - y)$, contrary to assumption. So we must be mistaken that $f(D(x - y))$ is open at all.

Now, suppose that $k = \mathbb{F}_q$ is a finite field. FINISH / IS IT TRUE? ☐

Exercise (3.1.11).

Proof. Let $B = k[x_0, \ldots, x_n]$ and $C = k[y_1, \ldots, y_n]$ with the usual gradings. Consider the map $\varphi : C \to B$ given by $y_i \mapsto x_i$. Note that this is a homomorphism of graded rings. Let C_+ be as usual and let $M = \varphi(C_+)B = (x_1, \ldots, x_n)$. Then by Lemma 2.3.40, φ induces a morphism $p : \operatorname{Proj}(B) \setminus V_+(M) \to \operatorname{Proj}(C)$. But $V_+(M) = \{z\}$, since $z = (1, 0, \ldots, 0)$ in homogeneous coordinates means that z corresponds to the prime ideal generated by $1x_i - 0x_0 = x_i$ for $1 \leq i \leq n$, and z is maximal among ideals not containing B_+.

In homogeneous coordinates, if we have a point $(a_0, \ldots, a_n)$, it corresponds to the ideal generated by all $a_i x_j - a_j x_i$, which contracts under φ to the ideal generated by all $a_i y_j - a_j y_i$ for $i, j \geq 1$, i.e. the point $(a_1, \ldots, a_n)$ in coordinates on $\mathbb{P}^{n-1}_k$. These computations work in any field and in particular to $\overline{k}$ after base-changing.

We will show that if X is a closed subset of $\operatorname{Proj} B$ not containing z, then it cannot contain the fiber $p^{-1}(y)$ for any $y \in \operatorname{Proj} C$. In this case, y is a prime ideal generated by homogeneous polynomials $f_i(y_1, \ldots, y_n)$ for some $1 \leq i \leq m$. Then, the ideal generated by $f_i(x_1, \ldots, x_n)$ in $\operatorname{Proj} B$ maps to y. If X were to contain this ideal, it would contain any prime containing it, since X is closed, but clearly each generator is in $(x_1, \ldots, x_n)$, which is precisely z. But then it is clear that this prime ideal is contained in the one generated by $x_1, \ldots, x_n$, which is exactly z. ☐

Exercise (3.2.1).

Proof. Note that Y has an open cover $\{\operatorname{Spec} A_i\}$ of finitely many affine schemes corresponding to Noetherian rings. Since $X \to Y$ is of finite type, each preimage of a $\operatorname{Spec} A_i$ is covered by finitely many affine opens corresponding to a finitely generated A_i-algebra. But a finitely generated algebra over a Noetherian ring is Noetherian, and the finite union of finite sets is finite, so we've covered X by finitely many Noetherian affine open subschemes, showing that X is Noetherian.

Since dimensions can be computed locally, the dimension of X is the supremum of the dimensions of each of the affine schemes in the above cover. But each one of these is finite, bounded by the dimension of A_i plus the number of generators, and there are only finitely many of them, so the supremum is also finite. ☐

Exercise (3.2.2).

Proof. Let $U \hookrightarrow X$ be an open immersion into the locally Noetherian scheme X. We can choose a cover of X by (not necessarily finitely many) Noetherian affines $\{\operatorname{Spec} A_i\}$. It suffices to show that each $U \cap \operatorname{Spec} A_i$ is covered by finitely many affines of finitely generated A_i-algebras. We drop the subscript in the sequel.

Since $U \cap \operatorname{Spec} A$ is open in $\operatorname{Spec} A$, it equals $\operatorname{Spec} A \setminus V(I)$ for some ideal I of A. Since A is Noetherian, we have $I = (f_1, \ldots, f_m)$ for some $f_i \in A$, whence

$$U \cap \operatorname{Spec} A = \operatorname{Spec} A \setminus V(I) = \operatorname{Spec} A \setminus V(f_1, \ldots, f_m) = \operatorname{Spec} A \setminus \bigcap_{i=1}^{m} V(f_i) = \bigcup_{i=1}^{m} D(f_i)$$

So, we're done, since $D(f_i) \cong \operatorname{Spec} A_{f_i}$ is affine and Noetherian since the localization of a Noetherian ring is Noetherian. □

Exercise (3.2.3).

Proof. Since $f : X \to Y$ is an immersion, we can factor it as $X \xrightarrow{i} Z \xrightarrow{j} Y$ with i an open immersion and j a closed immersion. We identify each scheme with its image, so that $X \subseteq Z \subseteq Y$, with i, j the inclusion maps on points. Then X is an open subset of Z, so it is of the form $U \cap Z$ for some open subset U of Y. Since $f(X) \subseteq U$, the map f factors through the open immersion $U \hookrightarrow Y$. I.e. we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ \downarrow g & & \downarrow j \\ U & \xrightarrow{u} & Y \end{array}$$

It now suffices to show that g is a closed immersion, and since closed immersions are stable under base change, it suffices to show that this diagram is Cartesian. So, to show that X satisfies the universal property, let W be a scheme with $v : W \to Z$ and $w : W \to U$ such that $j \circ v = u \circ w$. On points, we have $j(v(W)) = u(w(W)) \subseteq U$, so $v(W) \subseteq U \cap Z = X$. Since i is an open immersion, we get a unique map $x : W \to X$ such that $i \circ x = v$. Then

$$u \circ (g \circ x) = (u \circ g) \circ x = (j \circ i) \circ x = j \circ (i \circ x) = j \circ v = u \circ w$$

Since u is an open immersion, it is a monomorphism, so $g \circ x = w$. Hence w factors u and v as needed, and the uniqueness comes solely from uniquely factoring v, completing the argument.

Now, suppose f is quasi-compact and factors as $f = u \circ g$ for a closed immersion $g : X \to U$ and an open immersion $u : U \to Y$. As f is a quasi-compact, it also factors as $j \circ i$ for a map $i : X \to Z$ and the scheme-theoretic closure $j : Z \to Y$ of f. It suffices to show that i is an open immersion. On points, it is clear that i injects, since

$$i(a) = i(b) \implies u(g(a)) = j(i(a)) = j(i(b)) = u(g(b)) \implies g(a) = g(b) \implies a = b$$

So we have that U, Z are subsets of Y and we can identify X as a subset of $U \cap Z$. In fact, since g is a closed immersion, $X = U \cap W$ for some closed subset W of Y. But then X is contained in the closed subset $W \cap Z$ of Z and X is dense in Z, so $W \cap Z = Z$, i.e. $W \supseteq Z$. Hence $X = U \cap W \supseteq U \cap Z \supseteq X$, so these are all equal. This makes it clear that i is an open immersion of spaces, and it remains to see that $i^\#$ is an isomorphism.

Explicitly, we have $Z = V(\mathscr{I})$, where $\mathscr{I} = \ker(f^\#) = \ker(g^\# \circ u^\#)$. FINISH

Now, $f : X \to Y$ and $g : Y \to Z$ are immersions, so we can factor each as an open immersion followed by a closed immersion:

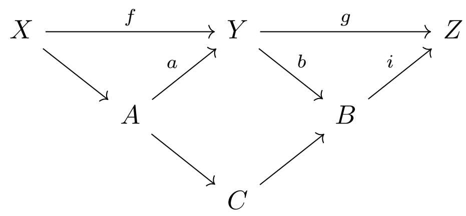
$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z \\ & \searrow \quad \nearrow^{a} & & \searrow^{b} \quad \nearrow^{i} & \\ & A & & B & \end{array}$$

Next, we'll show that $b \circ a$ is quasi-compact. The composition of quasi-compact morphisms is quasi-compact, so it suffices to show each one is quasi-compact. Closed immersions are quasi-compact, so we're done with a and only need to consider b. Then since i is also a closed immersion, it is a monomorphism, so the fiber product of two B-schemes is isomorphic to the fiber product of the same schemes over Z via $i : B \to Z$. In particular,

$$Y \cong Y \times_B B \cong Y \times_Z B$$

In particular, the map $b : Y \to B$ is the base change of $g : Y \to Z$ along i. But quasi-compactness is stable under base change, so b is quasi-compact as desired.

Now $b \circ a$ is quasi-compact and we've written it as a closed immersion followed by an open immersion. By the above, it is also an immersion, i.e. it factors as an open immersion followed by a closed immersion:



So we're done, since $X \to A \to C$ is an open immersion as the composition of open immersions, and $C \to B \to Z$ is a closed immersion as the composition of closed immersions. Hence, the overall composition $g \circ f$ is an immersion as claimed. ☐

Exercise (3.2.4).

Proof. First, note that x is the unique maximal ideal of $\mathscr{O}_{X,x}$, so let $y = f_x(x)$ and $s \in S$ be the point that x, y lie over. Choose a Noetherian affine neighborhood $\operatorname{Spec} R$ of s, and choose an affine open neighborhood $y \in \operatorname{Spec} A$ contained in the preimage of $\operatorname{Spec} R$. Then $f_x^{-1}(\operatorname{Spec} A)$ is an open subset of $\operatorname{Spec} \mathscr{O}_{X,x}$ containing x, whence it must be all of $\operatorname{Spec} \mathscr{O}_{X,x}$. So, f_x factors through $\operatorname{Spec} \mathscr{O}_{X,x} \to \operatorname{Spec} A$, which corresponds to a map $\varphi : A \to \mathscr{O}_{X,x}$. After choosing an affine neighborhood $\operatorname{Spec} B$ of $x \in X$, this becomes a map $\varphi : A \to B_x$. It would suffice to show that this lifts to a map $A \to B_f$ for some $f \notin x$. Indeed, if we can do this, then $D(f) \subseteq \operatorname{Spec} B$ is an open neighborhood of X containing x isomorphic to $\operatorname{Spec} B_x$, and the lift gives a map $D(f) \to \operatorname{Spec} A$. So, we're done with $U = D(f)$ after composing with $\operatorname{Spec} A \hookrightarrow Y$.

In other words, we've reduced to the algebraic problem of showing that for a Noetherian ring R, an R-algebra B, and a finitely generated R-algebra A, a map $\varphi_0 : A \to B_{\mathfrak{p}}$ extends to a map $\varphi : A \to B_f$. Since A is a finitely generated R-algebra, we can find a surjective map $\psi : R[T_1, \ldots, T_n] \to A$. Let $\varphi_0(\psi(T_i)) = f_i/g_i$ for some $f_i \in B$ and $g_i \notin \mathfrak{p}$. Then $R[T_1, \ldots, T_n]$ is Noetherian, so we can write $\ker(\psi) = (h_1, \ldots, h_m)$. Since

$$0/1 = \varphi_0(\psi(h_i(T_1, \ldots, T_n))) = h_i(f_1/g_1, \ldots, f_n/g_n) = a_i/b_i$$

for some $a_i \in B$ and $b_i \notin \mathfrak{p}$, we can write $a_i u_i = 0$ for some $u_i \notin \mathfrak{p}$.

Finally, write $g = \prod_{i=1}^n g_i$, $u = \prod_{i=1}^m u_i$, $f = gu$, and $F : R[T_1, \ldots, T_n] \to B_f$ defined by $F(T_i) = \left(u f_i \prod_{j \neq i} g_j\right)/f$. Then, for each $1 \leq i \leq m$:

$$\begin{aligned} F(h_i(T_1, \ldots, T_n)) &= h_i\left(u f_1 \left(\prod_{j \neq 1} g_j\right)/f, \ldots, u f_n \left(\prod_{j \neq n} g_j\right)/f\right) \\ &= (a_i s)/(b_i s) \\ &= 0 \end{aligned}$$

for some s such that $b_i s$ is a power of f, where the final equality follows from $a_i s f = a_i u_i \left(sg \prod_{j \neq i} u_j\right) = 0$. Hence $\ker(\psi) \subseteq \ker(F)$ and so it factors through it, i.e. we get a map $\varphi : A \to B_f$ with $F = \varphi \circ \psi$. If $\iota : B_f \to B_{\mathfrak{p}}$ is the localization map, then

$$\iota(\varphi(\psi(T_i))) = \iota(F(T_i)) = \left(u f_i \prod_{j \neq i} g_j\right)/f = f_i/g_i = \varphi_0(\psi(T_i))$$

So $\iota \circ \varphi \circ \psi = \varphi_0 \circ \psi$ since they agree on R and each T_i. But ψ is surjective, so $\iota \circ \varphi = \varphi_0$, which is what we wished to show. ☐

Exercise (3.2.5).

Proof. As above, this result follows from the affine case and gluing. In the affine case, we have a Noetherian ring R, a prime ideal $\mathfrak{p}$, finitely generated R-algebras A and B, and a map $\varphi_0 : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$. We seek $f \notin \mathfrak{p}$ such that there is a map $\varphi : A_f \to B_f$ factoring φ_0. FINISH ☐

Exercise (3.2.6).

Proof. Note, the claim does not make sense without an additional assumption. For example, when $X = \operatorname{Spec} \mathbb{F}_p$ and $Y = S = \operatorname{Spec} \mathbb{Z}$, with all maps defined by the universal property of $\operatorname{Spec} \mathbb{Z}$ being terminal, we indeed have a map of integral S-schemes that are finite-type over S, but it doesn't induce a map $\mathbb{F}_p = K(Y) \to K(X) = \mathbb{Q}$ since no such map exists. In particular, what's missing is that the map f needs to map the generic point of X to the generic point of Y; equivalently, we will assume henceforth that f is dominant.

Now, one direction is clear; if f induces an isomorphism between open subsets U of X and V of Y, then it also induces an isomorphism on the ring of sections, which gives the isomorphism on the corresponding fields of fractions, which is precisely the isomorphism $K(Y) \to K(X)$.

So, now suppose that $f_\xi : K(Y) \to K(X)$ is an isomorphism. Choose an open affine $\operatorname{Spec} R \subseteq S$ containing the image of ξ, choose an open affine $\operatorname{Spec} A \subseteq Y$ in the preimage of $\operatorname{Spec} R$ that also contains the image of ξ, and finally choose an open affine $\operatorname{Spec} B \subseteq X$ containing the preimage of $\operatorname{Spec} A$. We then have that A, B are domains and finitely generated R-algebras, and that f induces an R-algebra homomorphism $\varphi : A \to B$. The condition that f is dominant tells us that φ is injective, and f_ξ being an isomorphism tells us that φ induces an isomorphism between $\operatorname{Frac}(A)$ and $\operatorname{Frac}(B)$. After identifying A with its image, we assume $A \subseteq B$ and that $\operatorname{Frac}(A) = \operatorname{Frac}(B)$; it suffices to exhibit $B_h = A_g$ for some $g \in A$ and $h \in B$.

Note that since B is a finitely generated R-algebra, it is certainly a finitely generated A-algebra (say, with the same generators), so that $B = A[x_1, \ldots, x_n]$ for some $x_i \in B$. Each $x_i \in B$, so they are in $\operatorname{Frac}(B) = \operatorname{Frac}(A)$, so we may write them in the form $x_i = a_i/u_i$ for $a_i, u_i \in A$. Let $g = \prod_{i=1}^n a_i u_i$, and consider A_g, B_g. On the one hand, $A_g \subseteq B_g$, since B_g contains A and contains $1/g$. On the other hand, A_g contains A and

$$\frac{a_i^2 \prod_{j \neq i} a_j u_j}{g} = \frac{a_i}{u_i} = x_i$$

for each i, so A_g contains B. But A_g also obviously contains $1/g$, so it contains B_g. This completes the proof by taking $U = \operatorname{Spec} B_g$ and $V = \operatorname{Spec} A_g$. □

Exercise (3.2.7).

Proof. We will prove the claim in stages. First, assume that $\bar{X} = \operatorname{Spec} B, \bar{Y} = \operatorname{Spec} A$ are both affine. Then $A = \bar{k}[y_1, \ldots, y_r]/I$ and $B = \bar{k}[x_1, \ldots, x_s]/J$ for some ideals I, J. The canonical map $\bar{k}[y_1, \ldots, y_r] \twoheadrightarrow A$ gives a closed immersion $Y \hookrightarrow \mathbb{A}^r_{\bar{k}}$ and similarly we have a closed immersion $X \hookrightarrow \mathbb{A}^s_{\bar{k}}$. □

Exercise (3.2.8).

Proof. Suppose $f : X \to Y$ is a morphism of finite type with finite fibers, and let $y \in Y$. Then $X_y \cong f^{-1}(y)$ is a finite set, and the induced map $X_y \to k(y)$ is a morphism of finite type. For any $x \in X_y$, we can choose an open affine $\operatorname{Spec} A$ containing x, and this condition guarantees that A is a finitely generated $k(y)$-algebra. Then $\mathscr{O}_{X_y, x} = A_x$ is the localization of A at the prime x, which is a finitely generated $k(y)$-algebra, and so by Noether normalization is module-finite over a polynomial ring over $k(y)$ with as many variables as the dimension of A_x. So, we are done if $\dim A_x = 0$. But a polynomial ring in at least one variable over a field has infinitely many primes (Euclid), while A_x only has finitely many.

For a counterexample, consider a field extension L/K. Then the structure map $f : \operatorname{Spec} L \to \operatorname{Spec} K$ obviously has finite fibers, but in order for it to be quasi-finite, we need the stalk of $\operatorname{Spec} L$ at its only point to be finite over the residue field of $\operatorname{Spec} K$ at its point. I.e. we need L/K to be a finite field extension, so take L/K any infinite extension for a counterexample. □

Exercise (3.2.9).

Proof. Let $x \in X$ be a closed point. Then if $\operatorname{Spec} A$ is an open affine neighborhood of x, then A must be a finitely generated k-algebra, and we can compute the residue field $k(x)$ by localizing A at x and quotienting by the image of the maximal ideal. But localization commutes with quotients, so equivalently we can quotient and then localize. But since x is closed, it corresponds to a maximal ideal, so the quotient is a field, and the localization does nothing. That is, $k(x) \cong A/x$, where x on the right denotes the maximal ideal. But then $k(x)$ is a finitely generated k-algebra and a field, so by Zariski it is a finite field extension of k, and so embeds (in some way) into $\bar{k}$. This embedding gives a map $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k(x)$, which we can compose with the canonical map $\operatorname{Spec} k(x) \to X$ to get an element of $h \in X(\bar{k})$.

Now, by assumption, f and g induce the same map $X(\bar{k}) \to Y(\bar{k})$, and so $f \circ h = g \circ h$ (as morphisms of schemes). But $h(*) = x$, where $*$ is the unique point of $\operatorname{Spec} \bar{k}$, and so this gives $f(x) = f(h(*)) = g(h(*)) = g(x)$ as desired. I think there's an error in the next claim (that they agree on all points), unless we assume Y is separated over k. Roughly, the argument should be the analog of "f and g agree on a dense set and Y is Hausdorff, so $f = g$." In this case, the proper substitutions are that the set of closed points is indeed dense in X, and separatedness guarantees that therefore $f = g$. Indeed, then the argument is as follows:

Since Y/k is separated, $\Delta : Y \to Y \times_k Y$ is a closed immersion. Consider the base change along $X \to Y \times_k Y$ given by $x \mapsto (f(x), g(x))$ – that is, the morphism induced by the maps f and g – and denote it $u : K \to X$. Closed immersions are stable under base change, so u is a closed immersion. Hence, the image of u is a closed subset of X. On the other hand, I claim the image of u contains all points $x \in X$ such that $f(x) = g(x)$. Indeed, if $f(x) = g(x)$, then x lies over the pair $(f(x), g(x))$ in $Y \times_k Y$, and $\Delta(f(x)) = (f(x), f(x)) = (f(x), g(x))$ as well. But then there is a point of K that maps to $f(x) \in Y$ and $x \in X$, as desired.

Now, the image of u is a closed subset of X containing all closed points of X. So, we are done if the set of closed points is dense. It suffices to show this locally, so by passing to an open affine, we can assume $X = \operatorname{Spec} A$. The closure of the set of maximal ideals is $V(I)$ for some ideal I. This I must be contained in each maximal ideal, so

$$I \subseteq \bigcap_{\substack{\mathfrak{m} \in X \\ \mathfrak{m} \text{ is maximal}}} \mathfrak{m} = \sqrt{(0)}$$

since A is a finitely generated k-algebra. But then $V(I) \supseteq V(\sqrt{(0)}) = X$, so we're done.

Now, we would like to show that $f = g$ as morphisms of schemes. Again, it suffices to show this locally since we can then glue the morphisms back uniquely. So, for $x \in X$, choose an affine neighborhood $\operatorname{Spec} B$ of $f(x) = g(x)$ in Y, take the preimage to get an open subset of X, and reduce to an affine open neighborhood $\operatorname{Spec} A$ contained in this open set containing x. It suffices to show that $f = g$ when restricted to $\operatorname{Spec} A \to \operatorname{Spec} B$, and so we may assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Further, B is a finitely generated k-algebra, say $B = k[y_1, \dots, y_n]/I$ for some ideal I, which gives a closed immersion $\iota : \operatorname{Spec} B \hookrightarrow \mathbb{A}^n_k$. But closed immersions are monomorphisms, so showing that the compositions $\iota \circ f = \iota \circ g$ would imply that $f = g$. Hence, we may assume that $Y = \mathbb{A}^n_k$.

Finally, consider the base change to $\operatorname{Spec}(\bar{k})$. The maps $f_{\bar{k}}$ and $g_{\bar{k}}$ are equal by assumption, and so we have the following commutative diagram:

$$\begin{array}{ccc} X_{\bar{k}} & \longrightarrow & X \\ \downarrow & & \Big\downarrow g \Big\downarrow f \\ \mathbb{A}^n_{\bar{k}} & \longrightarrow & \mathbb{A}^n_k \end{array}$$

So, we would be done if the top arrow is an epimorphism. This is a map of affine schemes, so it also suffices to show that the corresponding map of rings is injective. But this is the map $A \to A \otimes_k \bar{k}$, which is injective since A is a free, and so flat, k-module. This completes the suggested reductions.

Now, we have two maps between affine schemes, which thus correspond to maps $\varphi, \psi : k[y_1, \dots, y_n] \to A$ for some finitely generated k-algebra A. We would like to show that the maps on Spec are equal, which will follow if $\varphi = \psi$ itself. Exercise 2.3.7 also tells us how to understand the maps on Spec for k-points, but since k is algebraically closed, every closed point is rational. So, we can conclude that for each maximal ideal $\mathfrak{m}$ of A, that $\varphi(y_i) + \mathfrak{m} = \psi(y_i) + \mathfrak{m}$ in the quotient $A/\mathfrak{m} \cong k$. I.e. $\varphi(y_i) - \psi(y_i) \in \mathfrak{m}$ for all i and all $\mathfrak{m}$. But again, A is a finitely generated k-algebra, so this shows that $\varphi(y_i) - \psi(y_i) \in \sqrt{(0)}$ for all i. But $X = \operatorname{Spec} A$ is geometrically reduced and $k = \bar{k}$, so A is reduced, whence $\sqrt{(0)} = (0)$. So $\varphi(y_i) = \psi(y_i)$ for all i, and so $\varphi = \psi$. □

Exercise (3.2.10).

Proof. Recall that we previously showed (exercise 2.3.20) that if A is a ring, G is a finite group of automorphisms of A, A^G is the invariant subring, and $p : \operatorname{Spec} A \to \operatorname{Spec} A^G$ the morphism induced by $A^G \hookrightarrow A$, then $p(x_1) = p(x_2)$ if and only if there is a $\sigma \in G$ such that $\sigma(x_1) = x_2$. In our case, G is the Galois group of K/k and $A = L \otimes_k K$; in order to show that G acts transitively on $\operatorname{Spec} A$, it thus suffices to show that $p(x_1) = p(x_2)$ for any $x_1, x_2 \in A$. So, we should compute $A^G = (L \otimes_k K)^G$. For this, consider the exact sequence

$$0 \to k \to K \xrightarrow{f} \prod_{\sigma \in G} K$$

of k-modules, where $f(a) = (\sigma(a) - a)_{\sigma \in G}$. Indeed, this is exact since the kernel of the nontrivial map is precisely those $a \in K$ such that $\sigma(a) = a$ for all σ; i.e. it is the fixed field of K under G, which is k by definition. Then, L is a free k-module, hence a flat k-module, so tensoring gives the exact sequence

$$0 \to L \to K \otimes_k L \xrightarrow{f \otimes \mathrm{id}_L} \left(\prod_{\sigma \in G} K \right) \otimes_k L$$

So, L is precisely the kernel of $f \otimes \mathrm{id}_L$. But this map acts exactly as G does: for a simple tensor $a \otimes b \in K \otimes_k L$, we have $\sigma(a \otimes b) = \sigma(a) \otimes b = (\sigma \otimes \mathrm{id}_L)(a \otimes b)$, and so the kernel of $f \otimes \mathrm{id}_L$ is precisely the fixed subring of $K \otimes_k L$. Note that this proof only requires L to be a k-module; nowhere did we use that it is a field.

But now, we're done, since $p(x_1) = p(x_2)$ is the unique point of $\operatorname{Spec} L$ for any x_1, x_2 as above.

Note that the action of G on X_K is as follows: each element $\sigma \in G$ induces a map which we also denote $\sigma : \operatorname{Spec} K \to \operatorname{Spec} K$. Thus we get a double Cartesian diagram

$$\begin{array}{ccccc} X_K & \xrightarrow{f_\sigma} & X_K & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\sigma} & \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} k \end{array}$$

The map f_σ is the action of σ on X_K. This description makes it clear that, on points, each f_σ is a homeomorphism with inverse $f_{\sigma^{-1}}$, and so carries irreducible components to irreducible components. So, it suffices to understand each f_σ on generic points of X_K.

First, let $\eta \in X$ denote its generic point. Then, I claim that each generic point of X_K lies in the fiber over η. Indeed, let $\xi \in X_K$ be a generic point of some irreducible component of X_K, and let $\pi : X_K \to X$ denote the projection. Choose an affine neighborhood $\operatorname{Spec} A$ of $\pi(\xi)$ and an affine neighborhood $\operatorname{Spec} B$ of ξ contained in $\pi^{-1}(\operatorname{Spec} A)$. Then $\eta \in \operatorname{Spec} A$ is the unique minimal prime, and $\xi \in \operatorname{Spec} B$ is one of its minimal primes. Note that $k \hookrightarrow K$ is free, hence flat, and that this is preserved by base change, so $A \to B$ is flat. So, this extension satisfies going down; in particular, $\pi(\xi)$ is a prime ideal in $\operatorname{Spec} A$, so it contains η, and if this containment were proper, we could find a prime ideal properly contained in ξ that projects to η. But ξ is minimal, so this cannot be. Hence, $\pi(\xi) = \eta$ as claimed.

So now, we know that all generic points lie in the fiber $(X_K)_\eta$ over η, which can be constructed itself as a fiber product via the following diagram:

$$\begin{array}{ccc} (X_K)_\eta & \longrightarrow & \operatorname{Spec} k(\eta) \\ \downarrow & & \downarrow \\ X_K & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} k \end{array}$$

But this makes it clear that $(X_K)_\eta \cong \operatorname{Spec}(k(\eta)) \times_k \operatorname{Spec} K = \operatorname{Spec}(k(\eta) \otimes_k K)$, and the first part of this problem demonstrates that G therefore acts transitively on it, as claimed. This holds for all Galois extensions K/k, and so it holds for k^{sep}/k, the separable closure of k in a fixed algebraic closure $\bar{k}$. Finally, the base change to $\bar{k}$ induces a homeomorphism $X_{\bar{k}} \to X_{k^{sep}}$, so they have the same irreducible components, which must therefore have equal dimension.

Note that there are only finitely many connected components of X_K since it is of finite type over a field. So, connected components are both open and closed. Let $U \subseteq X_K$ be a connected component of X_K, and define

$$Y = \bigcup_{\sigma \in G} \sigma(U)$$

Then, noting that σ is a homeomorphism, Y is open as the union of open sets and closed as the finite union of closed sets. So, writing $Z = X_K \setminus Y$ splits X_K as a disjoint union of closed sets. But now Y and Z are closed G-invariant subsets of X_K. I claim that this means that they are of the form $\pi^{-1}(Y'), \pi^{-1}(Z')$, respectively, for closed subsets $Y', Z' \subseteq X$, where $\pi : X_K \to X$ is the canonical projection. First, note that this claim would complete the proof. Indeed, $\operatorname{Spec} K \to \operatorname{Spec} k$ is surjective, so π is surjective. Hence, for $x \in X$, $x = \pi(t)$ for some $t \in X_K = Y \cup Z$. So, $x \in Y' \cup Z'$. Further, Y' and Z' are disjoint. Indeed, if $x \in Y' \cap Z'$, then the fiber over x intersects both Y and Z nontrivially. But G acts transitively on this fiber (by the same argument as above), so if one point of the fiber is in Y, every point of the fiber must be in Y since Y is G-invariant. Thus, we have written $X = Y' \sqcup Z'$ for closed Y' and Z', and Y' is nonempty since Y is nonempty, so by connectedness of X, we must have $Z' = \emptyset$, whence $Z = \emptyset$. So $Y = X_K$ and we see that all connected components are in a single orbit, as desired.

So, we are reduced to showing that if $Y \subseteq X_K$ is closed and G-invariant, then it is the preimage of some closed $Y' \subseteq X$. It suffices to show this when X is affine, since we can cover X by affine open neighborhoods, the G-orbit of a point of X_K is contained in the preimage of such a neighborhood, and being a preimage and closed can be computed locally. So, suppose $X = \operatorname{Spec} A$, whence $X_K = \operatorname{Spec}(A \otimes_k K)$, so $Y = V(I)$ for some ideal $I \subseteq A \otimes_k K$. WLOG, we can assume $I = \sqrt{I}$. We seek an ideal $J \subseteq A$ such that $V(I) = V(J(A \otimes_k K))$, and since I is radical, this is equivalent to $I = \sqrt{J(A \otimes_k K)}$. Choose

$J = I \cap A$, so one containtment is clear. For the other, note that if $x \in I$, then

$$\prod_{\sigma \in G} (T - \sigma(x)) \in A[x]$$

since each coefficient is in the fixed subring $(A \otimes_k K)^G = A$ (proven above). But since $x \in I$ and I is G-invariant, $\sigma(x) \in I$ for all σ, so that each coefficient is in I as well. So, each coefficient is in $I \cap A = J$, and since x is a root, we can solve for the leading term to get x^n as a sum of terms of the form $-a_i x^i$, which is in $J(A \otimes_k K)$. So $x \in \sqrt{J(A \otimes_k K)}$ as claimed, completing the argument. □

Exercise (3.2.11).

Proof. We'd like to show that $X_{\bar{k}}$ is connected, where $\bar{k}$ is a fixed algebraic closure of k. It suffices to show that $X_{k^{sep}}$ is connected, since these spaces are homeomorphic. If it isn't connected, we'd be able to find a finite subextension K such that X_K is disconnected, and without loss of generality, we can choose K/k to be a normal extension. So, it suffices to show that X_K is connected for all Galois extensions K/k. Let G be the galois group of K/k, and recall that we can canonically identify $X(K)^G$ with $X(k)$. So, since $X(k) \neq \emptyset$, we can find a point $x \in X(K)^G$. I claim that x must be contained in every connected component. Indeed, x is in some connected component U, and by the previous problem, G acts transitively on the connected components. So, if V is some other connected, component, then $\sigma(U) = V$ for some $\sigma \in G$ but $\sigma(x) = x$ since it is G-invariant, and so $x \in V$ as well. But we also know that distinct connected components are disjoint, so this is a contradiction unless X_K only has a single connected component. I.e. X_K must be connected, which is what we wished to show.

Note that this proof works almost verbatim for irreducibility, except that distinct irreducible components need not be disjoint. So, our counterexample should be one where all irreducible components contain a k-point of X. Consider $X = \operatorname{Spec} A$ where $A = \mathbb{R}[x, y]/(x^2 + y^2)$ and $k = \mathbb{R}$. Then X is irreducible since $x^2 + y^2$ is irreducible in $\mathbb{R}[x, y]$ and $X(k) \neq \emptyset$ since A has a maximal ideal with quotient k – namely, the ideal (x, y). However, taking $\bar{k} = \mathbb{C}$, we have $X_K = \mathbb{C}[x, y]/(x^2 + y^2) \cong \mathbb{C}[x, y]/(x + iy) \oplus \mathbb{C}[x, y]/(x - iy) \cong \mathbb{C}[x] \oplus \mathbb{C}[x]$, which is not irreducible. □

Exercise (3.2.12).

Proof. Throughout, fix an algebraic closure $\overline{K(X)}$ of $K(X)$, let $\bar{k}$ be the algebraic closure of k in $\overline{K(X)}$, and let k^{sep} be the separable closure of k in $\bar{k}$.

We are trying to compute the number of irreducible components of $X_{\bar{k}}$. First, we note that as usual, each generic point of $X_{\bar{k}}$ lies in the generic fiber of $\pi : X_{\bar{k}} \to X$. Indeed, if $\eta \in X$ is the generic point, and $\xi \in X_{\bar{k}}$ is some generic point, then choosing an affine neighborhood $\operatorname{Spec} A$ of $\pi(\xi)$, we get an affine neighborhood $\operatorname{Spec}(A \otimes_k \bar{k})$ of ξ. Since $k \hookrightarrow \bar{k}$ is free, it is flat, and so $A \to A \otimes_k \bar{k}$ is flat, whence it satisfies going-down. So, if $\pi(\xi)$ isn't minimal, then we can find a prime contained in ξ that does contract to a minimal prime of A, but ξ is minimal, so this is impossible. So $\pi(\xi)$ is a minimal prime, and η is the generic point, so it is contained in A and is the unique minimal prime there. So $\pi(\xi) = \eta$ as claimed.

Further, we have that the generic fiber is precisely composed of generic points. That is, if $\pi(\xi) = \eta$, then ξ is a generic point of $X_{\bar{k}}$. Indeed, choose affine neighborhoods as above, and note that $k \hookrightarrow \bar{k}$ is also integral, which means that $A \to A \otimes_k \bar{k}$ is also integral. Further, A is also a free k-module, so A is flat, which means this map is also an injection. So, $A \otimes_k \bar{k}$ also satisfies incomparability over A. In particular, if ξ were not a minimal prime of $A \otimes_k \bar{k}$, then it contains a minimal prime, but both of these must then contract to η, contradicting incomparability. So we must have that ξ is minimal, i.e. it is a generic point of $X_{\bar{k}}$.

So, we have reduced the problem to counting the size of the fiber over η. By considering the diagram

$$\begin{array}{ccccc}
\operatorname{Spec}(\bar{k} \otimes_k K(X)) & \longrightarrow & X_{\bar{k}} & \longrightarrow & \operatorname{Spec} \bar{k} \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \\
\operatorname{Spec} K(X) & \longrightarrow & X & \longrightarrow & \operatorname{Spec} k
\end{array}$$

we see that each square is cartesian, and so the fiber is isomorphic (as a scheme) to $\operatorname{Spec}(\bar{k} \otimes_k k(X))$, so we want to count this set.

To do this, let $L = k^{sep} \cap K(X)$. Then we have a tower of extensions $K(X)/L/k$, and we have that $\bar{k}$ is a k-scheme, so base-changing along both of these extensions gives a diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \operatorname{Spec} K(p) & & \\ \downarrow & & \downarrow & & \\ \operatorname{Spec}(\bar{k} \otimes_k K(X)) & \xrightarrow{f} & \operatorname{Spec}(\bar{k} \otimes_k L) & \longrightarrow & \operatorname{Spec} \bar{k} \\ \downarrow & & \downarrow & & \\ \operatorname{Spec} K(X) & \longrightarrow & \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} k \end{array}$$

We've also included in the diagram the fiber Y of f at some point $p \in \operatorname{Spec}(\bar{k} \otimes_k L)$. The reason for this is that we can compute the size of $\operatorname{Spec}(\bar{k} \otimes_k K(X))$ by summing over these fibers, so it suffices to find Y and sum over p. First, note that $K(p)$, the residue field at p, can be computed as the field of fractions of $(\bar{k} \otimes_k L)/p$, and so is the field of fractions of a finitely generated, algebraic, separable domain over $\bar{k}$. Hence it must be contained in $\bar{k}$ and of course, contains $\bar{k}$, i.e. it is $\bar{k}$. But then $Y = \operatorname{Spec}(K(X) \otimes_L \bar{k}) = \operatorname{Spec}(K(X) \otimes_L \bar{L})$. By Corollary 3.2.14(d), $K(X)$ is geometrically irreducible over L, so Y is irreducible. But also, $L \hookrightarrow \bar{k}$ is integral, so $K(X) \to K(X) \otimes_L \bar{k}$ is integral, and so $\dim(Y) = \dim(K(X)) = 0$. But then Y is irreducible and zero-dimensional, so it is a single point. So, finally, we've reduced to computing the size of $\operatorname{Spec}(L \otimes_k \bar{k})$.

This we can, at last, do explicitly. By the primitive element theorem, $L = k[T]/(f(T))$ for some irreducible separable polynomial $f \in k[T]$ of degree $n = [L : k]$. Thus,

$$L \otimes_k \bar{k} = (k[T]/f(T)) \otimes_k \bar{k} = \bar{k}[T]/(f(T)) \cong \bar{k}^n$$

where the last isomorphism comes from the chinese remainder theorem and the fact that f splits completely into distinct factors in $\bar{k}[T]$ by definition. So, it has exactly n prime ideals of the form

$$\bar{k}[T] \oplus \cdots \oplus \bar{k}[T] \oplus 0 \oplus \bar{k}[T] \oplus \cdots \oplus \bar{k}[T]$$

So, at last, we're done and have exactly $n = [L : k] = [K(X) \cap k^{sep} : k]$ irreducible components as claimed. □

Exercise (3.2.13).

Proof. We will show something slightly more general than the claim in this problem. Fix an algebraic extension L of k. Then I claim that if X_L is reduced, irreducible, or connected, then X_K is reduced, irreducible, or connected, respectively, for each subextension $L/K/k$; further, if the conclusion holds for each K with K/k finite, then the premise holds also.

Let's begin with the more straightforward directions. Let K be a subextension of L/k, and the corresponding base-change morphisms $X_L \to X_K \to X$. First, suppose that X_L is reduced. To show that X_K is reduced, it suffices to show this on an open cover. Let $\operatorname{Spec} A$ be an open affine of X_K, and note that then $\operatorname{Spec}(A \otimes_K L)$ is an open affine in X_L containing the preimage of $\operatorname{Spec} A$. So locally, the above morphism corresponds to the ring homomorphism $A \to A \otimes_K L$. But this map is injective since A is a (free, hence) flat K-algebra, and the codomain is reduced as it comes from an open subscheme of a reduced scheme, so A must be reduced as desired.

Now, suppose that X_L is irreducible. But $\operatorname{Spec} L \to \operatorname{Spec} K$ is surjective, which is preserved by base change, so $X_L \to X_K$ is surjective, and the image of an irreducible topological space is again irreducible. So X_K is irreducible as claimed. Similarly, if X_L is connected, then so is X_K since the image of a connected space is connected.

We'll now address each converse. Suppose first that X_L is not reduced. Again, this can be checked locally, so it must be the case that some stalk $\mathscr{O}_{X_L,p}$ is not a reduced ring. Choose an open affine $\operatorname{Spec} A \subseteq X$ containing the image of p, so that $\operatorname{Spec}(A \otimes_k L)$ is an open affine neighborhood of X_L containing p. Now $A \otimes_k L$ is not reduced (else each of its localizations would be reduced, including at p), so we can choose a nonzero nilpotent

$$f = \sum_{i=1}^{n} a_i \otimes b_i$$

for some n, some $a_1, \ldots, a_n \in A$, and some $b_1, \ldots, b_n \in L$. But now each b_i is algebraic over k by design, and there are finitely many of them, so $K = k[b_1, \ldots, b_n]$ is a finite extension of k. I claim that X_K is also not reduced. One of its affine neighborhoods is $\operatorname{Spec}(A \otimes_k K)$, but then this ring contains the element f. The map $A \otimes_k K \to A \otimes_k L$ is injective, so the fact that $f^N = 0$ in the latter ring means $f^N = 0$ in the former one as well, so that X_K is not reduced as claimed.

Second, suppose that X_L is not irreducible. If X itself is not irreducible, we are done, so assume WLOG that X is irreducible. Let W_1, W_2 be two distinct irreducible components of X_L, and endow them with the induced reduced subscheme structure. By Lemma 3.2.6, there exist finite subextensions K_1, K_2 of L/k and reduced closed subschemes $Z_1 \subseteq X_{K_1}$ and $Z_2 \subseteq X_{K_2}$ such that $W_1 = (Z_1)_L$ and $W_2 = (Z_2)_L$. As noted in the proof of that lemma, we can replace K_1, K_2 with any of their finite extensions contained in L, and so choosing, for example, the compositum allows us to assume $K_1 = K_2 = K$. But now again using that $\operatorname{Spec} L \to \operatorname{Spec} K$ is surjective, we get that $W_i \to Z_i$ is surjective, and so each Z_i is irreducible. Furthermore, I claim that each Z_i is actually an irreducible component of X_K, which would complete the proof of this converse, since they are distinct (having different base changes in X_L). Indeed, by similar reasoning to previous problems in this section, the generic points of X_L, X_K all make up the generic fiber. But the generic points of W_1, W_2 are generic points of X_L, so they map to the unique generic point in X, and they also map to the generic points of Z_1, Z_2. So, the generic points of Z_1, Z_2 lie in the generic fiber and so must be generic points of X_K, i.e. their closures in X_K are full irreducible components.

Finally, suppose that X_L is not connected. Then $\mathscr{O}_{X_L}(X_L)$ has a nontrivial idempotent f. But since $k \to L$ is flat and X is a Noetherian k-scheme, proposition 3.1.24 tells us that

$$\mathscr{O}_{X_L}(X_L) \cong \mathscr{O}_X(X) \otimes_k L$$

Under this isomorphism, we may write $f = \sum_{i=1}^n f_i \otimes b_i$ for some $f_1, \ldots, f_n \in \mathscr{O}_X(X)$ and $b_1, \ldots, b_n \in L$. As in the reduced case, we can then consider $K = k[b_1, \ldots, b_n]$ and conclude that

$$f \in \mathscr{O}_X(X) \otimes K \cong \mathscr{O}_{X_K}(X_K)$$

so that X_K is also disconnected. This completes the proof, and of course, taking $L = \bar{k}$ handles the original problem. □

Exercise (3.2.14).

Proof. Let $\mathfrak{P}$ be one of the properties reduced, integral, irreducible, connected. Let X be a geometrically $\mathfrak{P}$ k-variety and K/k an arbitrary field extension. To show that X_K is $\mathfrak{P}$, it suffices to show that $X_{\bar{K}}$ is $\mathfrak{P}$ for a fixed algebraic closure of K by the previous exercise. So, we may assume K is algebraically closed. Then K contains an algebraic closure $\bar{k}$ of k and $X_{\bar{k}}$ is also geometrically $\mathfrak{P}$, so we may also assume k is algebraically closed. When $\mathfrak{P} =$ reduced, the argument of the previous problem shows that we may also assume K is finitely generated over k, since a nilpotent arises from a finite sum (of course, we lose the assumption that K is algebraically closed in this case).

Let $\mathfrak{P} =$ reduced. Note now that we can write $K = k(a_1, \ldots, a_n)$ for some elements $a_i \in K$. Then K contains the subring $R = k[a_1, \ldots, a_n]$ and $Y = \operatorname{Spec}(R)$ is then an integral k-variety with $K(Y) = K$. Since Y is integral, it is reduced, and since k is algebraically closed, Y is geometrically reduced. So, by proposition 3.2.15, $K(Y) = K$ is a finite separable extension of a purely transcendental extension $k(T_1, \ldots, T_m)$. Now, to show that X_K is reduced, it suffices to show that base-changing to a simple transcendental extension preserves reducedness, since we already know that base-changing to an algebraic separable extension preserves reducedness. In other words, we want to show that $X_{k(t)}$ is reduced whenever X is reduced over an arbitrary field k.

For this, if $X_{k(t)}$ is not reduced, it fails to be reduced at some point, and projecting back to X, choosing a neighborhood, and pulling back converts the problem to showing that if A is a reduced finitely generated k-algebra, then $A \otimes_k k(t)$ is reduced. We can mimic the proof of Proposition 3.2.7. Namely, note that A is Noetherian, so it has finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, and so the injection $A \hookrightarrow \bigoplus_i (A/\mathfrak{p}_i)$ gives an injection $A \otimes_k k(t) \hookrightarrow \bigoplus_i (A/\mathfrak{p}_i) \otimes_k k(t)$. To show that the former is reduced, it suffices to show that the latter is reduced, for which is suffices to show each summand is reduced. Hence, we may assume that A is a domain. But then $A \otimes_k k(t)$ injects into $\operatorname{Frac}(A) \otimes_k k(t)$. But each element of this last ring is a fraction of the form $G(t)/g(t)$ with $G \in \operatorname{Frac}(A)[t]$ and $g \in k[t]$, so this is a domain, as a subring of $\operatorname{Frac}(A)(t)$, and so is itself reduced, as we wished to show.

Now, let $\mathfrak{P} =$ integral, so our reductions in the first paragraph allows us to assume only that k, and K are algebraically closed. Finish... □

Exercise (3.2.15).

Proof. □

Exercise (3.2.16).

Proof. □

Exercise (3.2.17).

Proof. □

Exercise (3.2.18).

Proof. □

Exercise (3.2.19).

Proof. □

Exercise (3.2.20).

Proof. □

Exercise (3.2.21).

Proof. □