Exercise (4.3.4). *Let k be a field. Let A be the polynomial ring $k[x, y]$. Show, by computing the dimension of the fibers, that the canonical morphism $\operatorname{Proj} A[T_0, T_1]/(yT_0 - xT_1) \to \operatorname{Spec} A$ is not flat.*

Proof. Note that $X = \operatorname{Proj} A[T_0, T_1]/(yT_0 - xT_1)$ and $Y = \operatorname{Spec} A$ are both k-varieties. We also have that $f : X \to Y$ is surjective. Indeed, let $\mathfrak{p} \in Y$ be a prime ideal of A. If $x \notin \mathfrak{p}$, then $\mathfrak{p}$ is the image of a point $\mathfrak{p}' \in \operatorname{Spec} k[x, y]_x \cong \operatorname{Spec} k[x, y, 1/x]$. But in X, we have $D_+(T_0) \cong \operatorname{Spec} A[T_1]/(y - xT_1) \cong \operatorname{Spec} k[x, y, T_1]/(y - xT_1) \cong \operatorname{Spec} k[x, y, y/x]$, and we have the inclusions $k[x, y] \subseteq k[x, y, y/x] \subseteq k[x, y, 1/x]$. This induces maps $\operatorname{Spec} k[x, y, 1/x] \to D_+(T_0) \to Y$, and the image of $\mathfrak{p}'$ is $\mathfrak{p}$, so we have found an element of X mapping to $\mathfrak{p}$. The argument is perfectly analogous if $y \notin \mathfrak{p}$.

Finally, suppose $x, y \in \mathfrak{p}$, so that $\mathfrak{p} = (x, y)$. Then writing $D_+(T_0)$ as $\operatorname{Spec} k[x, y, T_1]/(y - xT_1)$ makes it clear that $\mathfrak{p}k[x, y, T_1]/(y - xT_1)$ is prime, as it is prime in $k[x, y, T_1]$ and contains $y - xT_1$. Finally, it is clear that contracting back to $k[x, y]$ gives $\mathfrak{p}$, so we have shown surjectivity.

Hence, if f were a flat morphism, we would have that $\dim X_y = \dim X - \dim Y$ for all $y \in Y$. Directly,

$$\dim(X) = \dim(\operatorname{Proj} A[T_0, T_1]/(yT_0 - xT_1)) = \dim(\operatorname{Spec} k[x, y, T_0, T_1]/(yT_0 - xT_1)) - 1 = 3 - 1 = 2$$

and

$$\dim(Y) = \dim(\operatorname{Spec} k[x, y]) = 2$$

So, each fiber should be zero dimensional. On the other hand, the fiber over (x, y) contains at least $\operatorname{Spec} k[x, y, T_1]/(x, y) \cong \operatorname{Spec} k[T_1]$ by the above argument, which is 1-dimensional. So, f cannot be flat. $\square$