Numerical exploration in sphere packing, Fourier analysis, and physics

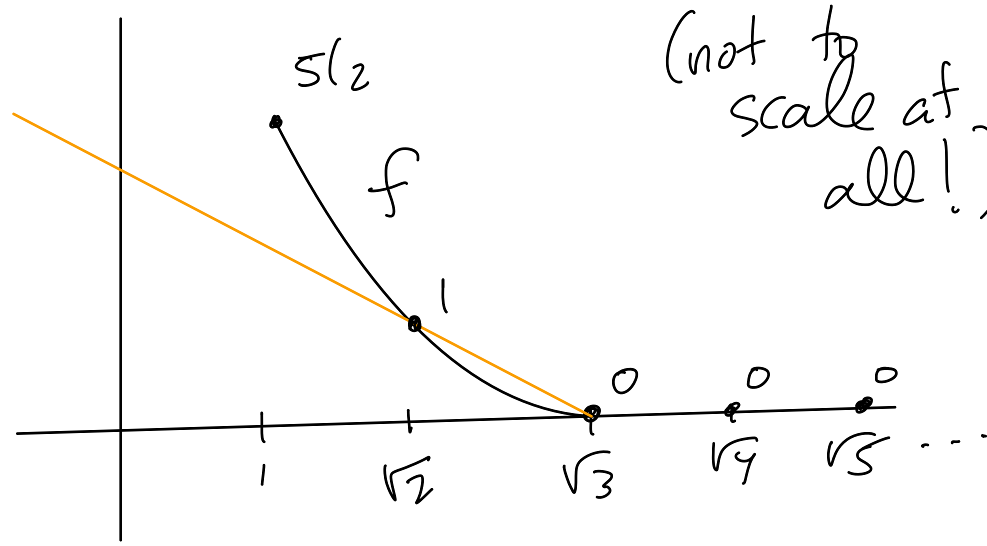
Lecture 4

Henry Cohn

PCMI 2022

Does E_8 minimize energy for every convex, decreasing potential fn.?

No:



(not to scale at all!)

Energy of E_8: $240 f(\sqrt{2}) + 2160 f(\sqrt{4}) + \cdots$
$= 240$

Energy of $\mathbb{Z}^8$: $16 f(1) + 112 f(\sqrt{2}) + 448 f(\sqrt{3}) + \cdots$
$= 152$

$152 < 240$

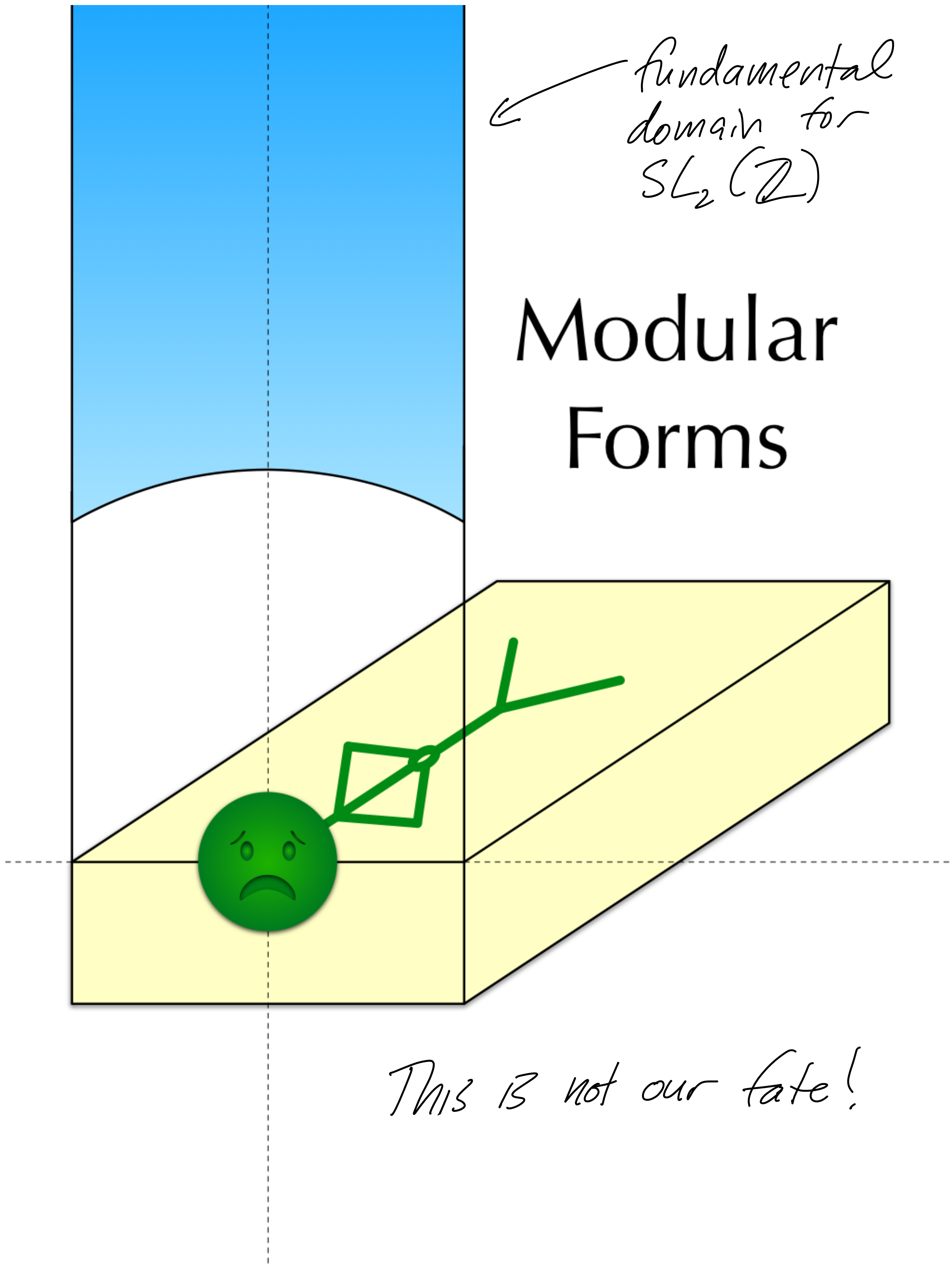
Today:

We'll use modular forms to interpolate a function.

When I first learned about modular forms, they were really intimidating.

If you're not already comfortable with them, don't worry. Lots of things are surprisingly concrete.

Modular Forms



This is not our fate!

$SL_2(\mathbb{R})$ acts on the upper half-plane

$$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$$

by linear fractional transformations.

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

$z \in \mathbb{H}$

$$\gamma z = \frac{az+b}{cz+d} \in \mathbb{H}$$

This is the conformal automorphism group of $\mathbb{H}$.

(also orientation-preserving isometries of the hyperbolic plane)

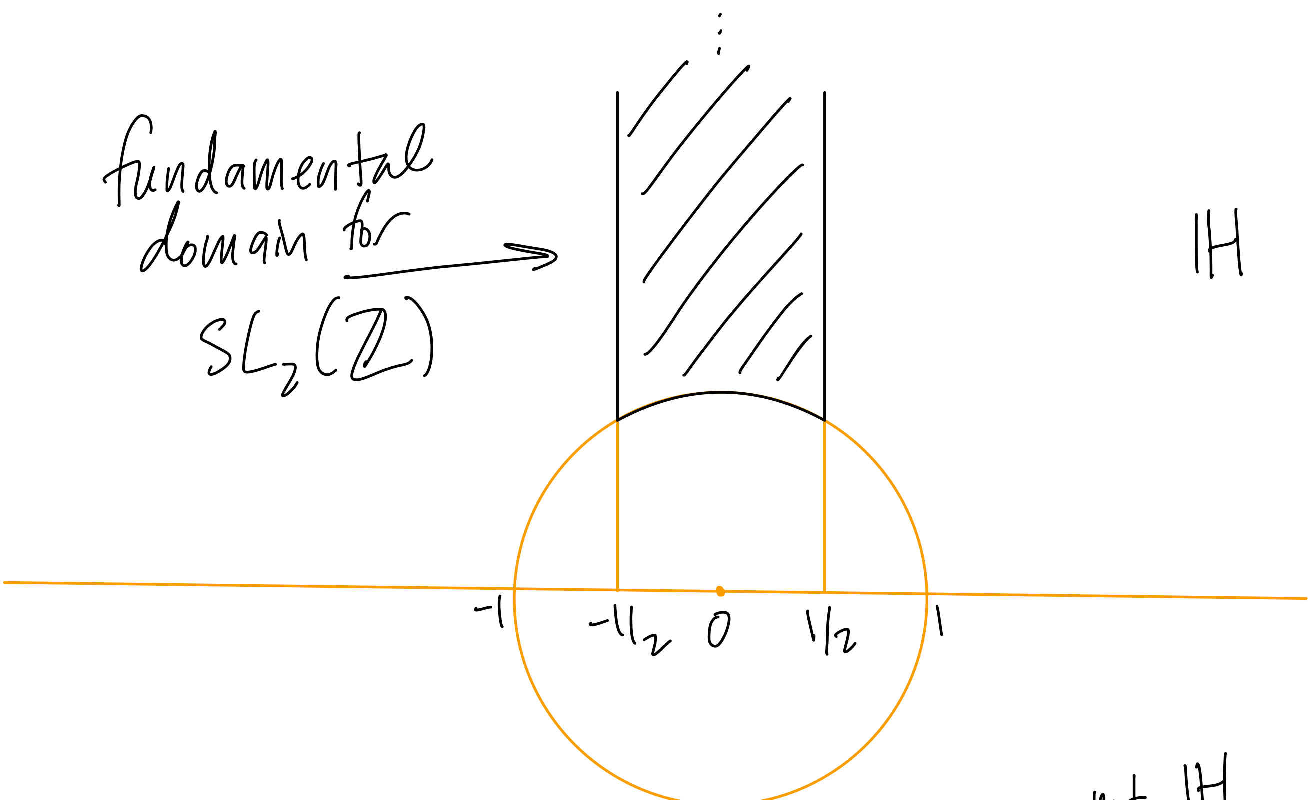
$SL_2(\mathbb{Z})$ is an important discrete subgroup of $SL_2(\mathbb{R})$.

It's generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -1/z$$

and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad z \mapsto z+1$$



A modular form for $SL_2(\mathbb{Z})$ of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

(1) $f|_k \gamma = f$ for all $\gamma \in SL_2(\mathbb{Z})$,

where

$$(f|_k \gamma)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and (2) $f(z)$ remains bounded as $\operatorname{Im} z \to \infty$ ("holomorphic at the cusp")

Why?

This theory is justified by examples and applications. It's not obviously compelling to a beginner.

These objects can be understood at many levels. For example, $f(z)\, dz^{\otimes k/2}$ is a section of the $k/2$ tensor power of the cotangent bundle of $SL_2(\mathbb{Z}) \backslash \mathbb{H}$.
(But this just raises the question of why we want such sections.)

In terms of S and T,

$$f(-1/z) = z^k f(z)$$

$$f(z+1) = f(z).$$

These functional equations come up surprisingly often.

The former is often deeper.

Example: theta series

$$\Theta_\Lambda(z) = \sum_{x \in \Lambda} e^{\pi i |x|^2 z}$$

converges for $z \in \mathbb{H}$

Think of it as a generating function for the vector lengths.

For $\Lambda = E_8$, $|x|^2 \in 2\mathbb{Z}$ for all x.

$$\Theta_{E_8}(z) = \sum_{n=0}^{\infty} N_n e^{2\pi i n z}$$

where $N_n = \#\{x \in E_8 : |x|^2 = 2n\}$.

$$\Theta_{E_8}(z) = \sum_{x \in E_8} e^{\pi i |x|^2 z}$$

$$= \sum_{n=0}^{\infty} N_n \, e^{2\pi i n z}$$

This implies $\Theta_{E_8}(z+1) = \Theta_{E_8}(z)$.

For S, we use Poisson summation.

$$E_8^* = E_8$$

Fourier transform of

$$x \longmapsto e^{\pi i z |x|^2}$$

Looks like S!

is

$$y \longmapsto \left(\frac{i}{z}\right)^{d/2} e^{\pi i (-1/z)|x|^2}$$

in $\mathbb{R}^d$

Here $d = 8$, $i^{d/2} = 1$.

Poisson:

$$\sum_{x \in E_8} e^{\pi i z |x|^2} = z^{-4} \sum_{y \in E_8} e^{\pi i (-1/z)|y|^2}$$

$$\Theta_{E_8}(z) = z^{-4}\,\Theta_{E_8}(-1/z)$$

So

$$\Theta_{E_8}|_4 S = \Theta_{E_8}$$

$$\Theta_{E_8}|_4 T = \Theta_{E_8}$$

$$\Theta_{E_8}(z) = 1 + 240 e^{2\pi i z} + \cdots$$

$\longrightarrow 1$ as $\mathrm{Im}\, z \to \infty$

modular form of weight 4

So our motivation is that we want to understand functions like the theta series of E_8 (weight 4) or Λ_{24} (weight 12).

Eisenstein series
(unfortunate notational clash w/ E_8)

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

(just a normalizing factor)

converges absolutely for $k>2$

vanishes when k is odd

$$E_k(z+1) = E_k(z)$$

$$E_k(-1/z) = z^k E_k(z)$$

(just from manipulating the series)

modular form of weight k
(for $k>2$)

Key fact: E_4 and E_6 generate the graded ring of modular forms for $SL_2(\mathbb{Z})$

E.g., $\Theta_{E_8} = E_4$

$$\Theta_{\Lambda_{24}} = \frac{7}{12} E_4^3 + \frac{5}{12} E_6^2$$

There are many other beautiful modular forms, e.g., Ramanujan's discriminant

$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$

Thanks to invariance under $z \mapsto z+1$, these functions all have Fourier series.

$$E_4(z) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) \, e^{2\pi i n z}$$

$$E_6(z) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) \, e^{2\pi i n z}$$

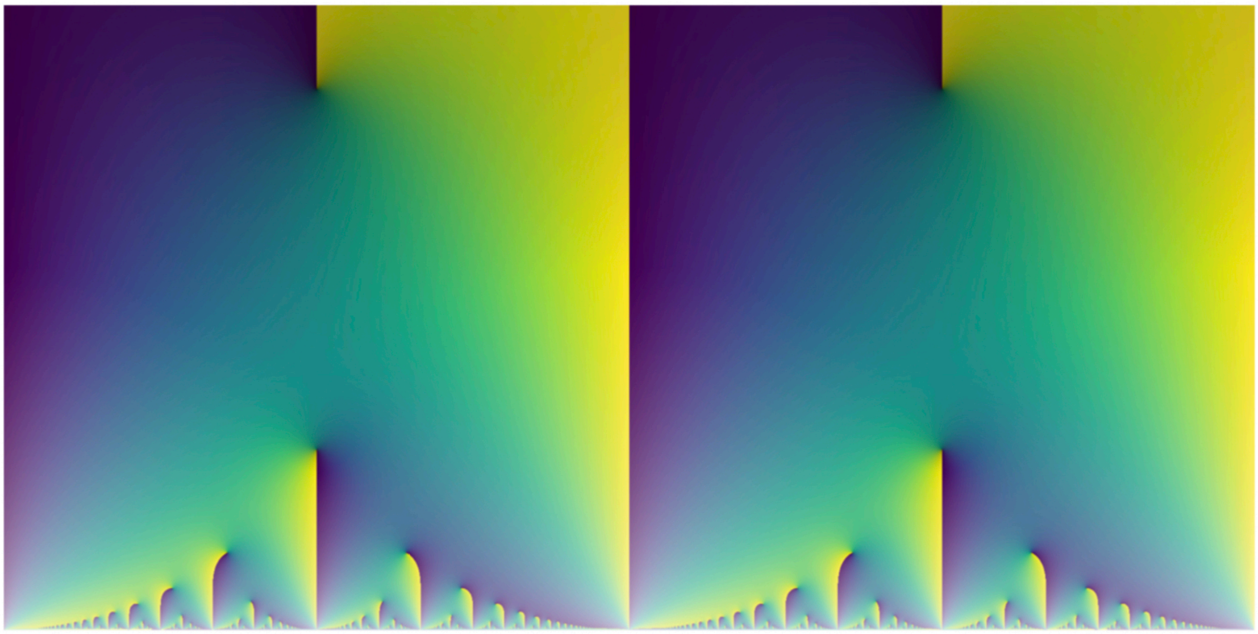
where $\sigma_k(n) = \sum_{d \mid n} d^k$.

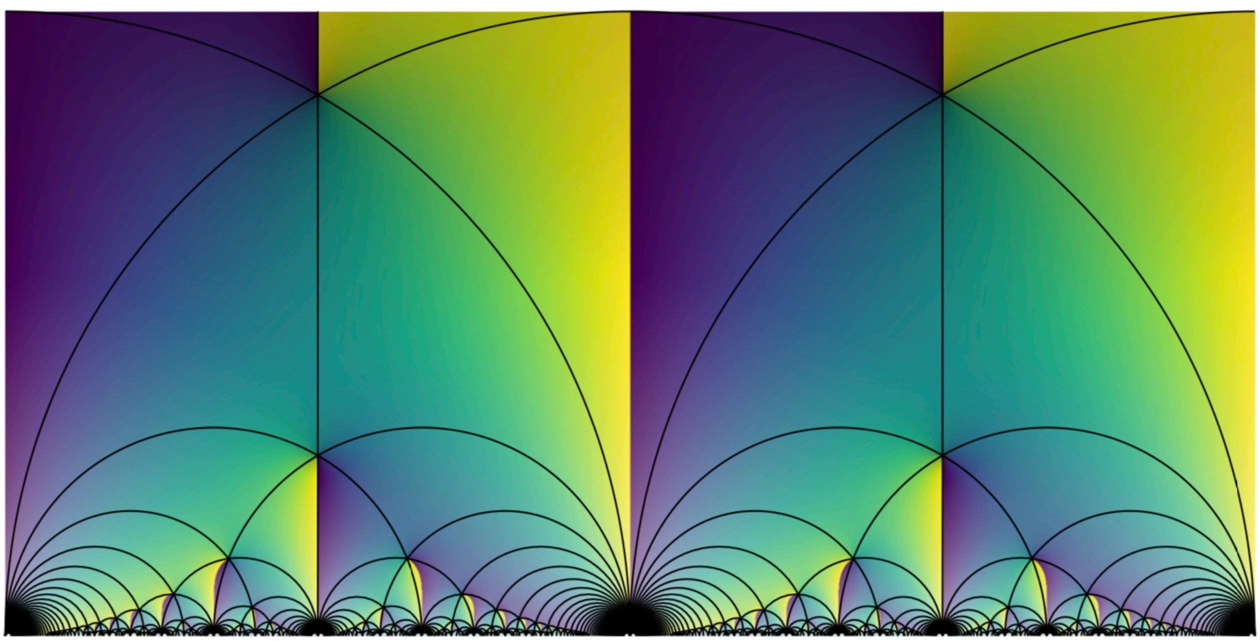
It's conventional to set $q = e^{2\pi i z}$.

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24}$$

vanishes at ∞, but $\Delta(z) \neq 0$ for $z \in \mathbb{H}$.

Plot of $E_4(z)$ for $-1 \leq \operatorname{Re} z \leq 1$
$0 < \operatorname{Im} z \leq 1$
(following Lowry-Duda)





(decorated w/ hyperbolic tiling)

We'll also need modular forms for the subgroup

$$\Gamma(2) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\}.$$

(i.e., $f|_k \gamma = f$ for $\gamma \in \Gamma(2)$, and $(f|_k \gamma)(z)$ bounded as $\operatorname{Im} z \to \infty$ for all $\gamma \in SL_2(\mathbb{Z})$)

Generated by weight 2 forms U, W:

$$U = (\Theta_{\mathbb{Z}})^4$$

$$W = U|_2 T$$

$$V = U - W$$

(many notations used)

$$U|_2 T = W \qquad V|_2 T = -V \qquad W|_2 T = U$$

$$U|_2 S = -U \qquad V|_2 S = -W \qquad W|_2 S = -V$$

Where are we going with this?

Let's interpolate radial Schwartz functions on $\mathbb{R}^8$.

We'll do 1^{st} order interpolation for simplicity: recover f from $f(\sqrt{n})$ and $\hat{f}(\sqrt{n})$ w/ $n \geq 2$.

(Radchenko and Viazovska, 2019)

We'll build only one function in an interpolation basis, as a taste.

We'll also look at -1 eigenfunctions of the Fourier transform.

Fourier eigenfunctions:

$$\hat{\hat{f}}(x) = f(-x) \text{ by}$$

Fourier inversion,

so $\hat{\hat{f}} = f$ for radial (or any even) f.

We can write

$$f = \underbrace{\frac{f - \hat{f}}{2}}_{-1 \text{ eigenfunction}} + \underbrace{\frac{f + \hat{f}}{2}}_{+1 \text{ eigenfunction}}$$

These cases turn out to require somewhat different techniques. We'll focus on -1 eigenfunctions.

Goal:

Construct a radial Schwartz function $g: \mathbb{R}^8 \to \mathbb{R}$

such that

$$\hat{g} = -g$$

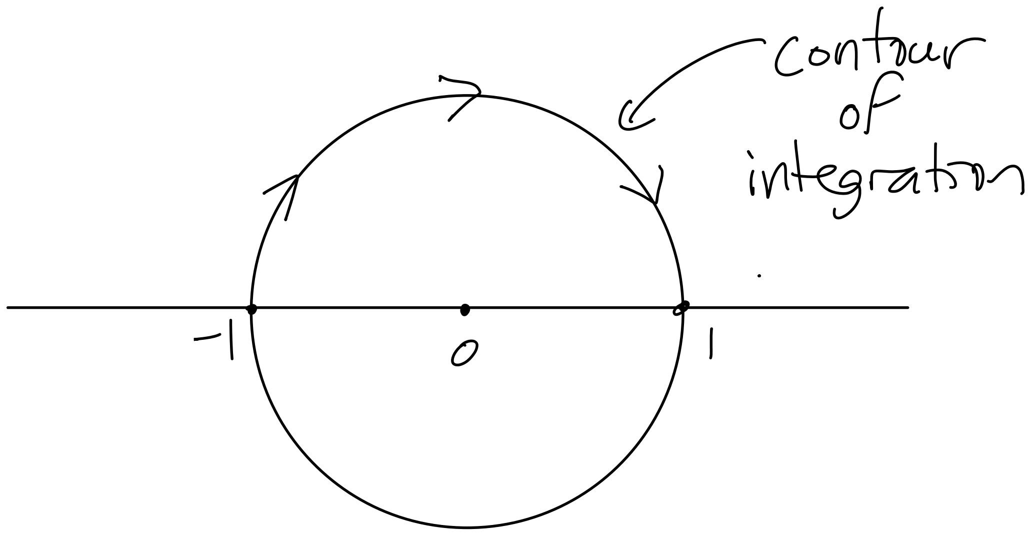
$$g(\sqrt{2}) = 1$$

$$g(\sqrt{n}) = 0 \quad \text{for } n \geq 3$$

(If we rotate the position of the 1 and analyze both eigenvalues ± 1, we'll get an interpolation basis.)

Construction:

$$g(x) = \frac{1}{2}\int_{-1}^{1} \varphi(z)\, e^{\pi i z |x|^2}\, dz$$



Which properties for φ will make this work?

$$g(x) = \frac{1}{2} \int_{-1}^{1} \psi(z)\, e^{\pi i z |x|^2}\, dz$$

Let's suppose $\psi(z+2) = \psi(z)$.

Fourier series

$$\psi(z) = \sum_{n \in \mathbb{Z}} a_n e^{\pi i n z}$$

$$g(\sqrt{n}) = \frac{1}{2} \int_{-1}^{1} \psi(z) e^{\pi i n z} dz$$

$= a_{-n}$ by orthogonality

We want $a_{-2} = 1$, $a_n = 0$ for $n < -2$.

$$g(x) = \frac{1}{2}\int_{-1}^{1} \psi(z)\, e^{\pi i z |x|^2}\, dz$$

$$\psi(z) = e^{-2\pi i z} + \sum_{n \geq -1} a_n e^{\pi i n z}$$

$\psi(z)$ should vanish as $z \to \pm 1$ to get a Schwartz function

When is g an eigenfunction of the Fourier transform?

$$g(x) = \frac{1}{2}\int_{-1}^{1} \psi(z)\, e^{\pi i z |x|^2}\, dz$$

8-dim'l Fourier transform of

$$x \longmapsto e^{\pi i z |x|^2}$$

is $\quad x \longmapsto z^{-4} e^{\pi i (-1/z)|x|^2}$

so

$$\hat{g}(x) = \frac{1}{2}\int_{-1}^{1} \psi(z)\, z^{-4} e^{\pi i (-1/z)|x|^2}\, dz$$

$$\hat{g}(x) = \frac{1}{2}\int_{-1}^{1} \psi(z)\, z^{-4}\, e^{\pi i (-1/z)|x|^2}\, dz$$

Let $u = -1/z$.

$$= \frac{1}{2}\int_{1}^{-1} \psi(-1/u)\, u^{4}\, e^{\pi i u |x|^2}\, \frac{du}{u^2}$$

$$= -\frac{1}{2}\int_{-1}^{1} \underbrace{u^2\, \psi(-1/u)}_{(\psi|_{-2} S)(u)}\, e^{\pi i u |x|^2}\, du$$

so $\hat{g} = -g$ if

$$\psi|_{-2} S = \psi$$

So φ should satisfy:

- $\varphi|_{-2} T^2 = \varphi$ (i.e., $\varphi(z+2) = \varphi(z)$)
- $\varphi|_{-2} S = \varphi$
- $\varphi(z) = e^{-2\pi i z} + \sum_{n \geq -1} a_n e^{\pi i n z}$
- φ vanishes at ± 1

Note: only meromorphic at ∞ and negative weight,

but $\Delta \cdot \varphi$ has weight 10 and is holomorphic at ∞.

$\Delta \cdot \psi$ is modular for $\Gamma(2)$
(a subgroup of the group generated by S, T^2)

linear combination of

$U^5, U^4W, U^3W^2, U^2W^3, UW^4, W^5$

and we know how S, T act on U, W

Simultaneous linear equations w/ unique sol'n

$$\psi = \frac{2U^4W - U^5}{\Delta}$$

$$\psi = \frac{2U^4 W - U^5}{\Delta}$$

$$= q^{-1} + 8q^{-1/2} - 240 - 6176 q^{1/2} - \cdots$$

$$\Longrightarrow g(\sqrt{n}) = \begin{cases} -240 & n = 0 \\ 8 & n = 1 \\ 1 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

as desired

What's left?

- a somewhat more elaborate integral for 2^{nd} order interpolation

- using modular forms for $\Gamma(2)$, quasimodular form E_2, and $\log \lambda$

This suffices to get each interpolation basis function.

Then

- systematize to solve for generating functions
- prove completeness