Numerical exploration in sphere packing, Fourier analysis, and physics

Exercises 1

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① a. Compute the volume of a ball in $\mathbb{R}^n$.

b. What is the density of the lattice packing $\mathbb{Z}^n$?

c. What is the density of the lattice packing

$$D_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \text{ is even} \right\} ?$$

d. How far from D_n is the point $(1, 0, \ldots, 0)$ in $\mathbb{R}^n$?

e. How far from D_n is the point $(1/2, 1/2, \ldots, 1/2)$ in $\mathbb{R}^n$?

f. Show that

$$E_8 := D_8 \cup \left(D_8 + (\overbrace{1/2, \ldots, 1/2}^{8})\right)$$

is a lattice.

g. What is the packing density of E_8?

(2) Define the <u>energy</u> of a lattice Λ under the potential fn. $r \mapsto e^{-\pi r^2}$ to be

$$\sum_{x \in \Lambda \setminus \{0\}} e^{-\pi |x|^2}.$$

a. What is the energy of a random lattice of determinant 1?

b. What is the energy of $\mathbb{Z}^n$? How does it compare w/ part a?

c. What's the lowest energy you can construct explicitly?

(3) Represent the symmetric matrix $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ by the point $(x, y, z) \in \mathbb{R}^3$.

a. What does the set of 2×2 positive-definite matrices look like geometrically in $\mathbb{R}^3$?

b. What are the facets of the Ryshkov polyhedron?

c. Find a vertex. Which facets does it lie on, and which edges?

d. Show that its neighbors along those edges are all equivalent under the action of $GL_2(\mathbb{Z})$.