Numerical exploration in sphere packing, Fourier analysis, and physics

Exercises 2

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(1) a. In a finite planar graph, let V be the # of vertices, E the # of edges, and F the # of faces in a planar embedding. Prove that $V-E+F=2$.

b. Prove that $3F \leq 2E$.

c. Prove that $E \leq 3V$. (Why does that mean the average degree is at most 6?)

d. Make sense of this for infinite graphs.

(2) Prove that $k \mapsto k \tan \frac{\pi}{k}$ is convex on $(1, \infty)$.

You can do this by remembering how to differentiate trig. functions, but is there a nicer proof? (I don't know offhand, but I hope so.)

(3) a. Verify the assertions about Laguerre polynomials in the notes.

b. What's the best polynomial $p(x)$ of degree 3 that you can find? What sort of bound do you get asymptotically as the dimension $n \to \infty$?

c. Can you numerically optimize the bound for low n and higher degrees? This is not so easy, but fun to play around with.