Rational points on varieties and the Brauer-Manin obstruction

Bianca Viray

University of Washington

My work appearing in this talk was predominantly completed on the lands of the **Coast Salish, Duwamish, Stillaguamish, and Suquamish** nations, & I am reporting on it from the lands of the **East Shoshone and Ute** nations.

$\alpha \in \mathrm{Br}\, X$

$$\begin{tikzcd}
& X(k) \arrow[r] \arrow[d, "x \mapsto x^*\alpha"] & X(\mathbb{A}_k) \arrow[d, "(x_v) \mapsto (x_v^*\alpha)"] \arrow[dr, "\varphi_\alpha"] & & \\
0 \arrow[r] & \mathrm{Br}\, k \arrow[r] & \bigoplus_v \mathrm{Br}\, k_v \arrow[r, "\sum \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$$X(\mathbb{A}_k)^\alpha := \varphi_\alpha^{-1}(0)$$

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{\alpha \in \mathrm{Br}\, X} X(\mathbb{A}_k)^\alpha$$

Approach: Embed $X(\mathbb{Q})$ into another set S that is more understandable/computable

Can we understand/compute $X(\mathbb{A}_k)^{\mathrm{Br}}$?

$\alpha \in \mathrm{Br}\, X$

$$\begin{tikzcd}
 & X(k) \arrow[r] \arrow[d, "x \mapsto x^*\alpha"] & X(\mathbb{A}_k) \arrow[d, "(x_v) \mapsto (x_v^*\alpha)"] \arrow[dr, "\varphi_\alpha"] & & \\
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Summary

Quadratic reciprocity (& higher order generalizations) carve out a (non-obvious) refined obstruction set

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$$

$\alpha \in \mathrm{Br}\, X$

$$\begin{tikzcd}
 & X(k) \arrow[r] \arrow[d, "x \mapsto x^*\alpha"] & X(\mathbb{A}_k) \arrow[d, "(x_v) \mapsto (x_v^*\alpha)"] \arrow[rd, "\varphi_\alpha"] & & \\
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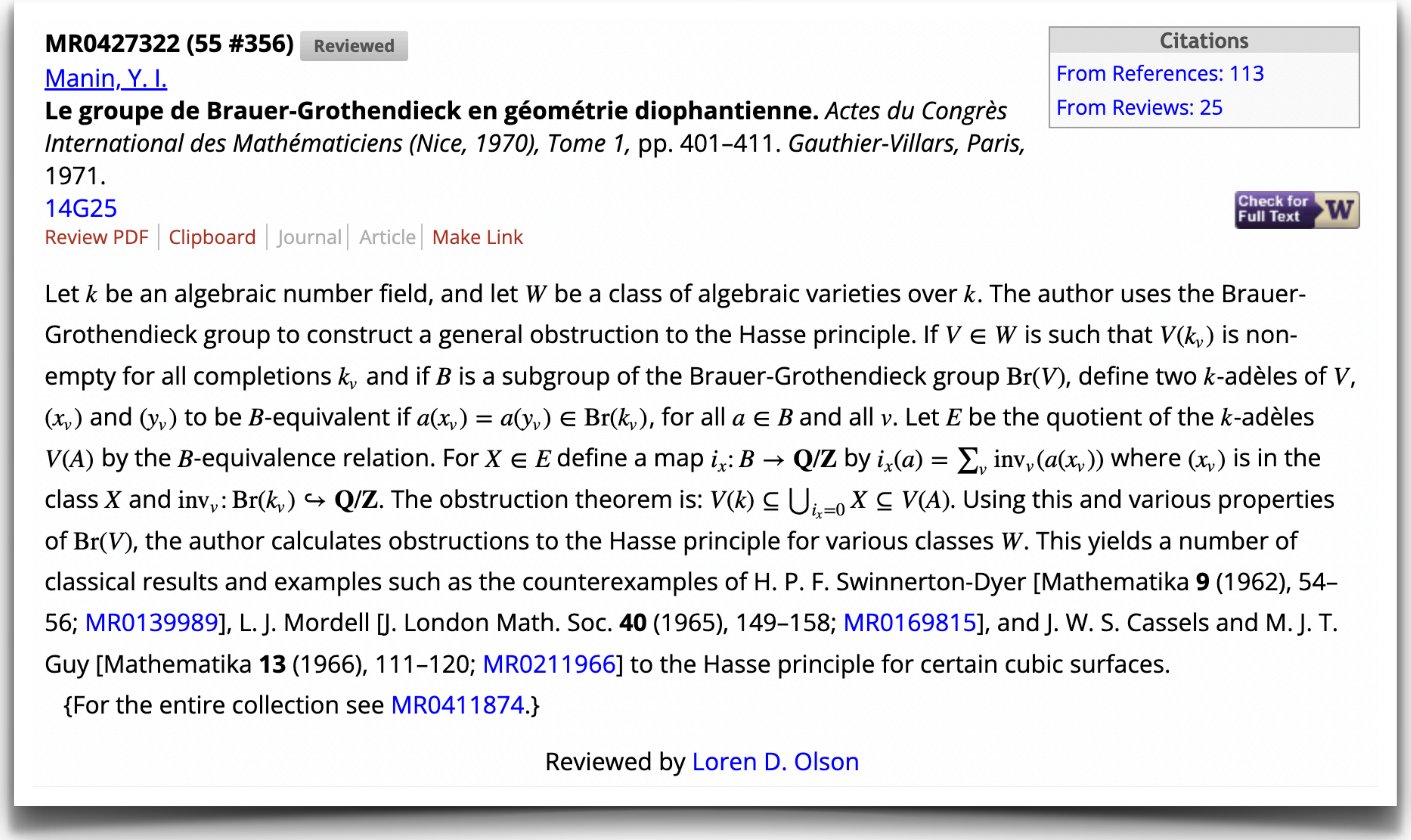


MR0427322 (55 #356) Reviewed

Manin, Y. I.

Le groupe de Brauer-Grothendieck en géométrie diophantienne. *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pp. 401–411. *Gauthier-Villars, Paris,* 1971.

14G25

Citations

From References: 113

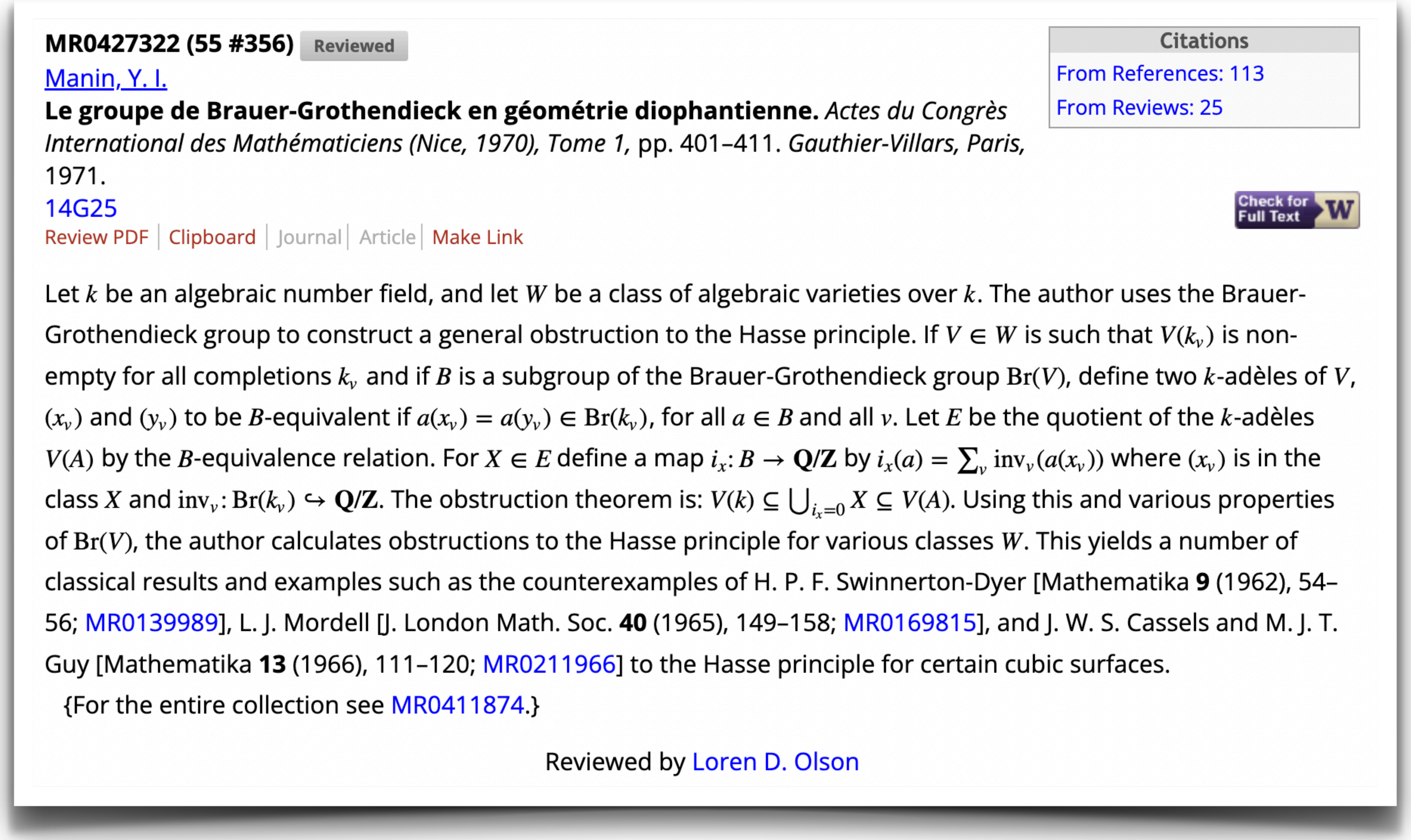
From Reviews: 25

Let k be an algebraic number field, and let W be a class of algebraic varieties over k. The author uses the Brauer-Grothendieck group to construct a general obstruction to the Hasse principle. If $V \in W$ is such that $V(k_v)$ is non-empty for all completions k_v and if B is a subgroup of the Brauer-Grothendieck group $\mathrm{Br}(V)$, define two k-adèles of V, (x_v) and (y_v) to be B-equivalent if $a(x_v) = a(y_v) \in \mathrm{Br}(k_v)$, for all $a \in B$ and all v. Let E be the quotient of the k-adèles $V(A)$ by the B-equivalence relation. For $X \in E$ define a map $i_x\colon B \to \mathbf{Q}/\mathbf{Z}$ by $i_x(a) = \sum_v \mathrm{inv}_v(a(x_v))$ where (x_v) is in the class X and $\mathrm{inv}_v\colon \mathrm{Br}(k_v) \hookrightarrow \mathbf{Q}/\mathbf{Z}$. The obstruction theorem is: $V(k) \subseteq \bigcup_{i_x=0} X \subseteq V(A)$. Using this and various properties of $\mathrm{Br}(V)$, the author calculates obstructions to the Hasse principle for various classes W. This yields a number of classical results and examples such as the counterexamples of H. P. F. Swinnerton-Dyer [Mathematika **9** (1962), 54–56; MR0139989], L. J. Mordell [J. London Math. Soc. **40** (1965), 149–158; MR0169815], and J. W. S. Cassels and M. J. T. Guy [Mathematika **13** (1966), 111–120; MR0211966] to the Hasse principle for certain cubic surfaces.

{For the entire collection see MR0411874.}

Reviewed by Loren D. Olson

Can we understand/compute $X(\mathbb{A}_k)^{\mathrm{Br}}$?



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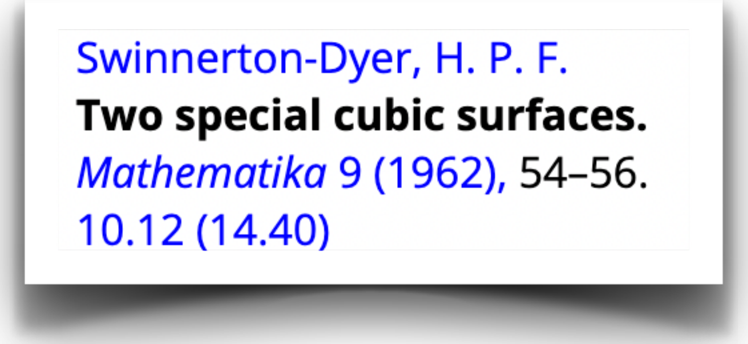

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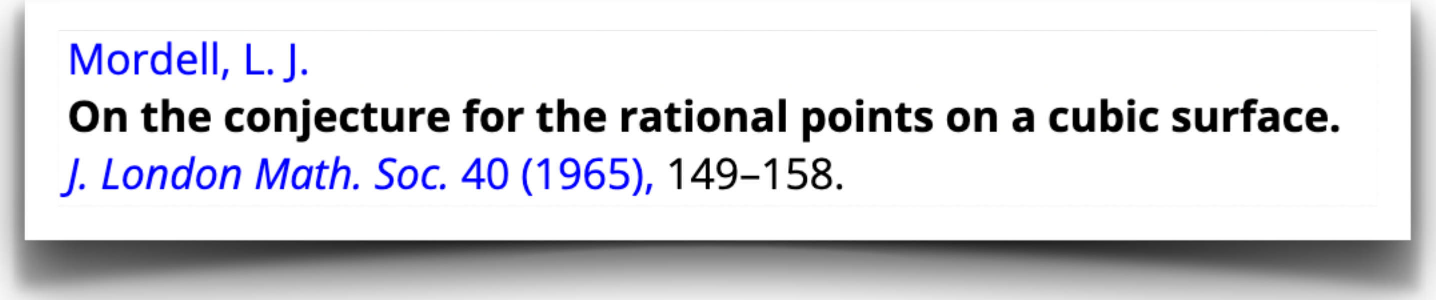

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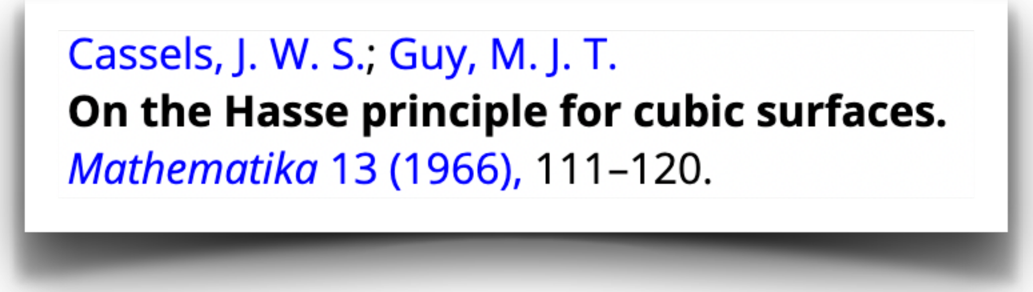

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☐ **MR3483120** Reviewed Newton, Rachel Transcendental Brauer groups of products of CM elliptic curves. *J. Lond. Math. Soc. (2)* 93 (2016), no. 2, 397–419. (Reviewer: Remke Kloosterman) 14F22 (11G05 14G25 14J28 14K15)
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An approach to understanding/computing $X(\mathbb{A}_k)^{\mathrm{Br}}$

An approach to understanding/computing $\mathrm{Br}\, X$

An approach to understanding/computing $\mathrm{Br}\ X$

Note: $\mathrm{Br}\ X$ is a torsion abelian group

An approach to understanding/computing $\mathrm{Br}\ X$

Define $\mathrm{Br}_0\ X := \mathrm{im}\left(\pi^*\colon \mathrm{Br}\ k \to \mathrm{Br}\ X\right)$

constant classes

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constant classes

$$\mathrm{Br}_1\ X := \ker\left(\mathrm{Br}\ X \to \mathrm{Br}\ \overline{X}\right)$$

algebraic classes

An approach to understanding/computing $\mathrm{Br}\ X$

Define $\mathrm{Br}_0\ X := \mathrm{im}\left(\pi^*\colon \mathrm{Br}\ k \to \mathrm{Br}\ X\right)$ *constant classes*

$\mathrm{Br}_1\ X := \ker\left(\mathrm{Br}\ X \to \mathrm{Br}\ \overline{X}\right)$ *algebraic classes*

(Exercise)

$X(\mathbb{A}_k)^{\mathrm{Br}_0\ X} = X(\mathbb{A}_k)$ and $X(\mathbb{A}_k)^{\mathrm{Br}}$ depends only on $\dfrac{\mathrm{Br}\ X}{\mathrm{Br}_0\ X}$

An approach to understanding/computing $\mathrm{Br}\ X$

Define $\mathrm{Br}_0\ X := \operatorname{im}\left(\pi^*\colon \mathrm{Br}\ k \to \mathrm{Br}\ X\right)$ *constant classes*

$\mathrm{Br}_1\ X := \ker\left(\mathrm{Br}\ X \to \mathrm{Br}\ \overline{X}\right)$ *algebraic classes*

$$0 \to \frac{\mathrm{Br}_1\ X}{\mathrm{Br}_0\ X} \to \frac{\mathrm{Br}\ X}{\mathrm{Br}_0\ X} \to \frac{\mathrm{Br}\ X}{\mathrm{Br}_1\ X} \to 0$$

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Define $\mathrm{Br}_0\ X := \mathrm{im}\left(\pi^*\colon \mathrm{Br}\ k \to \mathrm{Br}\ X\right)$ *constant classes*

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An approach to understanding/computing Br X

$$0 \to \frac{\mathrm{Br}_1 X}{\mathrm{Br}_0 X} \to \frac{\mathrm{Br}\, X}{\mathrm{Br}_0 X} \to \frac{\mathrm{Br}\, X}{\mathrm{Br}_1 X} \to 0$$

$$\frac{\mathrm{Br}_1 X}{\mathrm{Br}_0 X} \xrightarrow{\sim} \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X})$$

exact

$$0 \to \frac{\mathrm{Br}\, X}{\mathrm{Br}_1 X} \to (\mathrm{Br}\, \overline{X})^{\mathrm{Gal}(\bar{k}/k)} \to \mathrm{H}^2(k, \mathrm{Pic}\, \overline{X})$$

An approach to understanding/computing $X(\mathbb{A}_k)^{\mathrm{Br}}$

(assuming we've computed $\mathrm{Br}\ X/\mathrm{Br}_0\ X$)

Let $\alpha \in \mathrm{Br}\ X$ and let v be a place of k.

An approach to understanding/computing $X(\mathbb{A}_k)^{\mathrm{Br}}$

(assuming we've computed $\mathrm{Br}\ X/\mathrm{Br}_0\ X$)

Let $\alpha \in \mathrm{Br}\ X$ and let v be a place of k.

Then $\mathrm{ev}_\alpha \colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is locally constant,

An approach to understanding/computing $X(\mathbb{A}_k)^{\mathrm{Br}}$

(assuming we've computed $\mathrm{Br}\ X/\mathrm{Br}_0\ X$)

Let $\alpha \in \mathrm{Br}\ X$ and let v be a place of k.

Then $\mathrm{ev}_\alpha \colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is locally constant,
and if $\alpha_{k_v^{\mathrm{ur}}} = 0$, then ev_α is constant.

An approach to understanding/computing $X(\mathbb{A}_k)^{\mathrm{Br}}$

(assuming we've computed $\operatorname{Br} X/\operatorname{Br}_0 X$)

Let $\alpha \in \operatorname{Br} X$ and let v be a place of k.

Then $\mathrm{ev}_\alpha\colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is locally constant,
and if $\alpha_{k_v^{\mathrm{ur}}} = 0$, then ev_α is constant.

However, "unless there is a reason why not",

often have $\operatorname{im} \mathrm{ev}_\alpha = \operatorname{ord}(\alpha_v)^{-1}\mathbb{Z}/\mathbb{Z}$ for some v.

An approach to understanding/computing $X(\mathbb{A}_k)^{\mathrm{Br}}$

(assuming we've computed $\mathrm{Br}\ X/\mathrm{Br}_0\ X$)

Let

WARNING: Examples in the literature (and in the exercises) are **rigged** so that computations simplify!

However, "unless there is a reason why not",

often have $\operatorname{im} \mathrm{ev}_\alpha = \operatorname{ord}(\alpha_v)^{-1}\mathbb{Z}/\mathbb{Z}$ for some v.

Upshot

Computing $X(\mathbb{A}_k)^{\mathrm{Br}}$ can be doable,
but is often HARD.

Upshot

Computing $X(\mathbb{A}_k)^{\mathrm{Br}}$ can be doable,
but is often HARD.

What does the theory tell us about why this is hard?

there is no free lunch

idiom

Save Word

Definition of *there is no free lunch*

—used to say that it is not possible to get something that is desired or valuable without having to pay for it in some way

No Free Lunch

or

Brauer classes **want** to obstruct adelic points

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Unless there is a reason otherwise,
should expect $X(\mathbb{A}_k)^{\alpha} \subsetneq X(\mathbb{A}_k)$.

Brauer classes **want** to obstruct adelic points

Unless there is a reason otherwise,
should expect $X(\mathbb{A}_k)^\alpha \subsetneq X(\mathbb{A}_k)$.

Theorem[Harari '94]

Given a family of varieties $\mathscr{V} \to \mathbb{A}^1$,
with $\operatorname{Br}\mathscr{V}$ trivial, but $\operatorname{Br}\mathscr{V}_\eta$ nontrivial and [...],
$\exists\infty\ t_0 \in \mathbb{A}^1(k)$, such that $\mathscr{V}_{t_0}(\mathbb{A}_k)^{\operatorname{Br}} \subsetneq \mathscr{V}_{t_0}(\mathbb{A}_k)$.

Brauer classes **want** to obstruct adelic points

Unless there is a reason otherwise,
should expect $X(\mathbb{A}_k)^\alpha \subsetneq X(\mathbb{A}_k)$.

Theorem[Bright '15]

Let $\alpha \in \operatorname{Br} X$ & $v \nmid \operatorname{ord}(\alpha)$ be such that
$\partial_v(\alpha)$ has order n, $\#\mathbb{F}_v > > 0$ and [...],
then $\operatorname{ev}_{\alpha_v}$ has image $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Brauer classes **want** to obstruct adelic points

Unless there is a reason otherwise,
should expect $X(\mathbb{A}_k)^\alpha \subsetneq X(\mathbb{A}_k)$.

Theorem[Pagano, following Bright-Newton]

There exists a K3 surface X with a place of good reduction v and an $\alpha \in \operatorname{Br} X$ such that $\operatorname{ev}_{\alpha_v}$ is non-constant and $X(\mathbb{A}_k)^\alpha \subsetneq X(\mathbb{A}_k)$.

Brauer classes **want** to obstruct adelic points

Unless there is a reason otherwise,
should expect $X(\mathbb{A}_k)^\alpha \subsetneq X(\mathbb{A}_k)$.

When computing $X(\mathbb{A}_k)^{\mathrm{Br}}$,
cannot bypass computing ev_{α_v} for all
$\alpha \in \mathrm{Br}\, X_v / \ker(\mathrm{Br}\, X_v \to \mathrm{Br}\, X^{\mathrm{ur}})$

From Lecture 1

Given a variety $X/\mathbb{Q}$, how do we *prove* that $X(\mathbb{Q}) = \varnothing$?

Goal: Compute $X(\mathbb{A}_k)^{\mathrm{Br}}$

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From Lecture 1

Given a variety $X/\mathbb{Q}$, how do we *prove* that $X(\mathbb{Q}) = \varnothing$?

Goal: ~~Compute $X(\mathbb{A}_k)^{\mathrm{Br}}$~~

When computing $X(\mathbb{A}_k)^{\mathrm{Br}}$,
cannot bypass computing ev_{α_v} for all
$\alpha \in \mathrm{Br}\, X_v/\ker(\mathrm{Br}\, X_v \to \mathrm{Br}\, X^{\mathrm{ur}})$

From Lecture 1

Given a variety $X/\mathbb{Q}$, how do we *prove* that $X(\mathbb{Q}) = \varnothing$?

Goal: determine whether $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$.

Goal: determine whether $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$.

Why is this simpler?

Goal: determine whether $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$.

Why *may* this be simpler?

Goal: determine whether $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

Why *may* this be simpler?

X projective

Goal: determine whether $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$.

Why *may* this be simpler?

X projective $\Rightarrow X(\mathbb{A}_k)$ compact

Goal: determine whether $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

Why *may* this be simpler?

X projective $\Rightarrow$ $X(\mathbb{A}_k)$ compact $\Rightarrow$

$$\begin{gathered} X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset \\ \Downarrow \\ X(\mathbb{A}_k)^{B} = \emptyset, \end{gathered}$$

for some finite B

Let $\mathscr{B} \subset \operatorname{Br} X$

$\mathscr{B}$ **captures** the Brauer-Manin obstruction if

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Examples:

- C deg. d genus 1 curve
  $\Rightarrow (\mathrm{Br}\, C/\mathrm{Br}_0 C)[d^\infty]$ completely captures
- [Swinnerton-Dyer '99]
  X cubic surface $\Rightarrow (\mathrm{Br}\, X/\mathrm{Br}_0 X)[3]$ completely captures.
- [Colliot-Thélène, Poonen '00]
  X cubic or quartic del Pezzo $\Rightarrow \exists \alpha \in \mathrm{Br}\, X$ that captures.

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Then $X(\mathbb{A}_k)^{\mathrm{Br}_1}$ is effectively computable.
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$$\frac{\operatorname{Br}_1 X}{\operatorname{Br}_0 X} \xrightarrow{\sim} \mathrm{H}^1(k, \operatorname{Pic} \overline{X})$$

exact

$$0 \to \frac{\operatorname{Br} X}{\operatorname{Br}_1 X} \to (\operatorname{Br} \overline{X})^{\operatorname{Gal}(\bar{k}/k)} \to \mathrm{H}^2(k, \operatorname{Pic} \overline{X})$$

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$$\begin{tikzcd} {} \arrow[d, squiggly] \\ {} \end{tikzcd}$$

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when $i = 1$, used in descent on abelian varieties

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Methods often already start by computing $X(\mathbb{A}_k)^{\mathscr{B}}$ for such $\mathscr{B}$.

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[Creutz, Viray '18]

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May be hard to say anything about a capturing subgroup *a priori*.

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[Creutz, Viray , Voloch '18] C curve of genus ≥ 2

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The open
(questions)

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del Pezzo surfaces ($1 \leq \deg \leq 9$) or conic bundles

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[Corn '07] $(\operatorname{Br} X/\operatorname{Br}_0 X)[2^\infty]$ is isomorphic to

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What collection of elements are necessary?

Rational surfaces

del Pezzo surfaces ($1 \leq \deg \leq 9$) or conic bundles

X conic bundle

$\operatorname{Br} X[2]$ generates $\operatorname{Br} X / \operatorname{Br}_0 X$

$\operatorname{Br} X / \operatorname{Br}_0 X$ can be arbitrarily large

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Does Br $[6^\infty]$ capture? Is there a general negative result?

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Where is the dividing line?