Rigid cocycles
& Real quadratic singular moduli (Part III)

PCMI Summer School 2022

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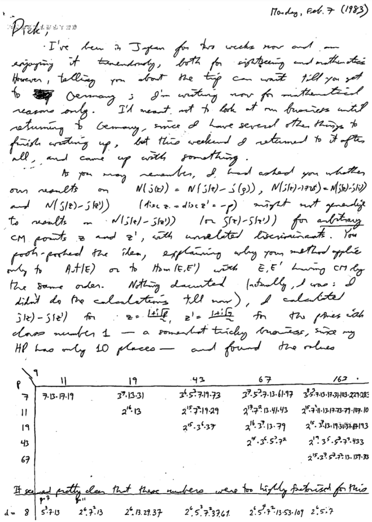
Zagier's letter (February 1983)



Monday, Feb. 7 (1983)

Dick,

I've been in Japan for two weeks now and am enjoying it tremendously, both for sightseeing and mathematics. However, telling you about the trip can wait till you get to Germany; I'm writing now for mathematical reasons only. I'd meant not to look at our business until returning to Germany, since I have several other things to finish writing up, but this weekend I returned to it after all, and came up with something.

As you may remember, I had asked you whether our results on $N(j(z)) = N(j(z) - j(\rho))$, $N(j(z)-1728) = N(j(z)-j(i))$ and $N(j(z)-j(z'))$ (disc z = disc $z' = -p$) might not generalize to results on $N(j(z)-j(z'))$ (or $j(z)-j(z')$) for arbitrary CM points z and z', with unrelated discriminants. You pooh-poohed the idea, explaining why your method applies only to $Aut(E)$ or to $Hom(E,E')$ with E, E' having CM by the same order. Nothing daunted (actually, I was: I didn't do the calculations till now), I calculated $j(z)-j(z')$ for $z = \frac{1+i\sqrt{d}}{2}$, $z' = \frac{1+i\sqrt{d'}}{2}$ for the primes with class number 1 — a somewhat tricky business, since my HP has only 10 places — and found the values

p \ q	11	19	43	67	163
7	$7\cdot13\cdot17\cdot19$	$3^7\cdot13\cdot31$	$3^6\cdot5^3\cdot7\cdot19\cdot73$	$3^7\cdot5^3\cdot7\cdot13\cdot61\cdot97$	$3^8\cdot5^3\cdot7\cdot13\cdot17\cdot31\cdot103\cdot229\cdot283$
11		$2^{16}\cdot13$	$2^{15}\cdot7^2\cdot19\cdot29$	$2^{17}\cdot7^2\cdot13\cdot41\cdot43$	$2^{15}\cdot7^2\cdot11\cdot13\cdot17\cdot73\cdot79\cdot107\cdot10$
19			$2^{15}\cdot3^6\cdot37$	$2^{16}\cdot3^7\cdot13\cdot79$	$2^{15}\cdot3^7\cdot13\cdot19\cdot31\cdot37\cdot67\cdot193$
43				$2^{15}\cdot3^6\cdot5^3\cdot7^2$	$2^{17}\cdot3^5\cdot5^3\cdot7^3\cdot433$
67					$2^{15}\cdot3^7\cdot5^3\cdot7^2\cdot13\cdot139\cdot33$

It seemed pretty clear that these numbers were too highly factorised for this

	p=7	p=11				
d = 8	$5^3\cdot7\cdot13$	$2^6\cdot7^2\cdot13$	$2^6\cdot13\cdot29\cdot37$	$2^6\cdot5^3\cdot7^2\cdot37\cdot61$	$2^6\cdot5^4\cdot7^2\cdot13\cdot53\cdot109$	$2^6\cdot5^3\cdot7$

One of the examples discussed in this letter is

$$j\left(\frac{1+\sqrt{-7}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right)$$

$$= -3^3\cdot5^3 \ + \ 2^{18}\cdot3^3\cdot5^3\cdot23^3\cdot29^3$$

$$= \ 3^8\cdot5^3\cdot7\cdot13\cdot17\cdot31\cdot103\cdot229\cdot283$$

Zagier then tabulates the prime factorisations of the positive integers N_x of the form

$$N_x = (7\cdot163 - x^2)/4$$

x	N_x	x	N_x	x	N_x
1	$3\cdot5\cdot19$	13	3^5	25	$3\cdot43$
3	283	15	229	27	103
5	$3^2\cdot31$	17	$3\cdot71$	29	$3\cdot5^2$
7	$3\cdot7\cdot13$	19	$3\cdot5\cdot13$	31	$3^2\cdot5$
9	$5\cdot53$	21	$5^2\cdot7$	33	13
11	$3\cdot5\cdot17$	23	$3^2\cdot17$		

Zagier observes that all prime divisors q satisfy

$$\left(\frac{-7}{q}\right) \neq 1, \quad \left(\frac{-163}{q}\right) \neq 1.$$

The work of Gross–Zagier

Zagier: conjectures an explicit formula for

$$\operatorname{ord}_q \ \operatorname{Nm} j(\tau_1) - j(\tau_2)$$

as a character sum, proves it using the real analytic Eisenstein family of Hecke (1924)

$$E_s(z_1, z_2) \qquad \text{over } \mathbf{Q}(\sqrt{D_1 D_2}).$$

Remark: He does *not* use CM theory.
He challenges Gross to find algebraic proof.

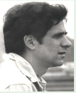
Gross: for any occurring q the reduction of endomorphism rings gives injections

$$\alpha_1, \alpha_2 : \mathcal{O}_{D_1}, \mathcal{O}_{D_2} \hookrightarrow R \subset B_{\infty q}.$$

Counts the intersection multiplicity:

$$\operatorname{ord}_q = \sum_{\substack{[\alpha_1],[\alpha_2] \\ b \in \Gamma_1 \backslash R_1^\times / \Gamma_2}} m_q \cdot (\alpha_1 \frown b\alpha_2 b^{-1})$$

where images coincide $\pmod{q^{m_q - 1}}$.



Gave rise to important later developments, for instance:

- *Birch–Swinnerton-Dyer conjecture.* Gross–Zagier (1986) and Gross–Kohnen–Zagier (1987) study height pairings of Heegner points on modular Jacobians:

$$\begin{aligned} L'(E,1)L(E_D,1) &\doteq h(P_D), \\ L'(E,1)\int_{D_1 D_2} f_E(z)dz &\doteq h(P_{D_1}, P_{D_2}). \end{aligned}$$

- *Traces of singular moduli.* Zagier (2002) shows generating series is modular, weight 3/2.

RM values of rigid cocycles

Last time, we introduced rigid cocycles $Z^1(\Gamma, \mathcal{M}^\times)$ modelled on the rational cocycles defined by Knopp (1978) and its multiplicative lifts, where

$$\Gamma := \mathrm{SL}_2(\mathbf{Z}[1/p]) \circlearrowleft \mathcal{M}^\times := \text{ Meromorphic functions on } \mathcal{H}_p$$

Input:

- a pair $(F, G) \in \mathcal{F}_{D_1} \times \mathcal{F}_{D_2}$ with $D_1, D_2 > 0$ non-squares,
- a prime p such that $(D_1/p) = (D_2/p) = -1$,

Output:

- the invariant $\Theta_p^\times[F, G] := \Theta_F^\times[G] \in \mathbf{P}^1(\mathbf{C}_p)$.

Joint with Darmon

Essential part is the (polynomial time in m) computation mod p^m of the product

$$\Theta_F^\times(S) \doteq \prod_{Q \in \Sigma_F^p} (z - r(Q))^{\mathrm{sgn}\, Q(0)} \quad \text{where } \Sigma_F^p := \{\langle a, b, c\rangle \in F \cdot \Gamma \ : \ ac < 0\}$$

Example: Let $p = 2$ and $F = \langle 1, -1, -1\rangle$ of $D_1 = 5$. Compute $\Theta_2^\times[F, G] \pmod{2^{300}}$ for forms $G = \langle 1, 3, -3\rangle$ and $\langle -1, 3, 3\rangle$ of $D_2 = 21$. Using LLL we find that they are roots of

$$91x^4 + 112x^3 + 123x^2 + 112x + 91, \qquad \text{roots in } \mathbf{Q}(\sqrt{-3}, \sqrt{-35})$$

Algebraic (!). Hilbert class fields are $H_1 = \mathbf{Q}(\sqrt{5})$ and $H_2 = \mathbf{Q}(\sqrt{-3}, \sqrt{-7})$.

Now choose $p = 3$ and quadratic forms

$$F = \langle 1, -1, -1 \rangle \quad D_1 = 5$$
$$G = \langle 1, 6, -2 \rangle \quad D_2 = 44$$

$\Theta_3^\times[F, G] \pmod{3^{200}}$ is root of polynomial (spl. field $K = \mathbf{Q}(\sqrt{5}, \sqrt{11}, \sqrt{-1})$)

$$48841x^8 + 115280x^6 + 164562x^4 + 115280x^2 + 48841$$

Note $48841 = 13^2 \cdot 17^2$, and tabulate the positive integers $N_x := (D_1 D_2 - x^2)/4$:

x	N_x	x	N_x	x	N_x
0	$5 \cdot 11$	6	$2 \cdot 23$	12	19
2	$2 \cdot 3^3$	8	$3 \cdot 13$	14	$2 \cdot 3$
4	$3 \cdot 17$	10	$2 \cdot 3 \cdot 5$		

Observation 1: Only get primes in K above non-split q dividing a positive integer

$$\frac{D_1 D_2 - x^2}{4p}$$

What is the influence of the choice of prime p? Compute $\Theta_p^\times[F, G]$ for

$$F = \langle 1, 3, -1 \rangle \quad D_1 = 13,$$

For $D_2 = 12$ we find

$p = 5$	$p = 7$
$\frac{1 \pm 4\sqrt{-3}}{7}$	$\frac{3 \pm 4\sqrt{-1}}{5}$

For $D_2 = 45$ we find

$p = 2$	$p = 7$
$\frac{150824917 \pm 100674475\sqrt{-3}}{2 \cdot 7^2 \cdot 13 \cdot 37 \cdot 67 \cdot 73}$	$\frac{1 \pm \sqrt{-15}}{2^2}$

For $D_2 = 108$ we find

$p = 5$	$p = 7$
$\frac{1237487 \pm 857860\sqrt{-3}}{7^2 \cdot 19 \cdot 31 \cdot 67}$	$\frac{128 \pm 2046\sqrt{-1}}{2 \cdot 5^2 \cdot 41}$

Observation 2: There is a relation

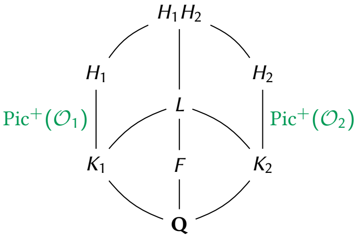
$$\mathrm{ord}_{\mathfrak{p}}\, \Theta_q^\times[F, G] \;\leftrightarrow\; \mathrm{ord}_{\mathfrak{q}}\, \Theta_p^\times[F, G].$$

Factorisation conjecture

If $D_1, D_2 > 0$ coprime discriminants and p inert for both, then for $i \in \{1,2\}$ we define

$$K_i \supset \mathcal{O}_i := \mathbf{Z}\left[\frac{D_i+\sqrt{D_i}}{2}\right]$$
$$H_i := \text{Narrow ring class field } \mathcal{O}_i$$
$$G_i := \mathrm{Gal}(H_i/\mathbf{Q}) = \mathrm{Pic}(\mathcal{O}_i) \rtimes \langle \mathrm{Frob}_p \rangle$$

When $\Theta_p^\times[F,G]$ is algebraic, should be in H_1H_2.

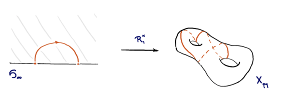


Intersections on Shimura curves

Choose optimal $\alpha_i : \mathcal{O}_i \hookrightarrow R \subset B_{pq}$ indefinite quat. alg. over $\mathbf{Q}$ of conductor pq. The $R_1^\times$-conjugacy classes $[\alpha_i]$ admit a free action of G_i. The q-intersection number

$$\mathrm{Int}_q(\alpha_1,\alpha_2) \quad := \sum_{b \in \Gamma_1 \backslash R_1^\times / \Gamma_2} m_q \cdot (\alpha_1 \frown b\alpha_2 b^{-1}).$$

is a q-weighted version of the intersection product of the classes $[\alpha_i] \in \mathrm{H}_1(X_{pq}, \mathbf{Z})$ on $X_{pq} = \mathcal{H}_\infty / R_1^\times$, the Shimura curve associated with B_{pq}.



Let $G := \mathrm{Gal}(H_1 H_2/\mathbf{Q}) \simeq \mathrm{Gal}(H_1/\mathbf{Q}) \times \mathrm{Gal}(H_2/\mathbf{Q}) = G_1 \times G_2$.

Factorisations

Choose a prime $\mathfrak{q} \mid (q)$ in $H_1 H_2$. Define

$$\mathrm{F}_{p,q} := \sum_{g \in G} \mathrm{ord}_{\mathfrak{q}^g} \left(\Theta_p^\times[F, G] \right) \cdot [g] \in \mathbf{Z}[G]$$

Intersections

Choose optimal $\alpha_i : \mathcal{O}_i \hookrightarrow R$. Define

$$\mathrm{I}_{p,q} := \sum_{g=(g_1,g_2) \in G} \mathrm{Int}_q(\alpha_1^{g_1}, \alpha_2^{g_2}) \cdot [g] \in \mathbf{Z}[G]$$

Conjecture: $\mathrm{F}_{p,q} = \mathrm{I}_{p,q}$ in $\mathbf{Z}[G]/G$.

Why should we believe this conjecture? Want to test it! Algorithm by James Rickards.

Example. Let $(D_1, D_2) = (13, 136)$ and $p = 7$. We have $\mathrm{Pic}^+(\mathcal{O}_2) \simeq C_4 = \langle s \rangle$.
We recognise $\Theta_7^\times[F, G] \pmod{7^{300}}$ as root of the polynomial (with splitting field $H_1 H_2$)

$$36673x^8 - 77740x^7 + 82758x^6 - 93080x^5 + 113387x^4 - 93080x^3 + 82758x^2 - 77740x + 36673$$

Note: Factorisation of constant coefficient $36673 = 7 \cdot 13^2 \cdot 31$.

James Rickards computes that the only non-trivial $\mathrm{I}_{7,q}$ are:

q	$\mathrm{I}_{7,q}$
7	$(1 + [t])(1 - [s^2])$
13	$(1 + [t])(1 - [s] + [s^2] - [s^3])(1 + [r])$
31	$(1 + [t])(1 - [s^2])$

What happens for larger primes p?

Try for instance $p = 11$ and $F = \langle 1, 3, -3\rangle$ of $D_1 = 13$. We attempt as usual

$$\text{For } G = \langle 1, 4, -4\rangle \qquad \mathrm{LLL}\left(\Theta_{11}^{\times}[F, G]\right) = \text{Unconvincing...} \quad \notin \overline{\mathbf{Q}}?$$

Suspected 'culprit' is lifting obstruction to $\mathrm{H}^1(\Gamma, \mathcal{M}^{\times}) \to \mathrm{H}^1(\Gamma, \mathcal{M}^{\times}/\,\mathbf{C}_p^{\times})$, lives in

$$\mathrm{H}^2(\Gamma, \mathbf{C}_p^{\times}) \simeq \mathrm{H}^1(\Gamma_0(p), \mathbf{C}_p^{\times}) \simeq \left(\mathbf{C}_p^{\times}\right)^{2g+1} \qquad g := \mathrm{genus}(X_0(p)).$$

Solution 1: Kill it

For $p = 11$ it is killed by Hecke operator $(W_p - I)$. We find that

$$\Theta_{11}^{\times}[F \cdot W_p, G]/\Theta_{11}^{\times}[F, G]$$

computed $\pmod{11^{50}}$ satisfies

$$13x^4 + 12x^3 + 14x^2 + 12x + 13,$$

with splitting field $\mathbf{Q}(\sqrt{-1}, \sqrt{-3})$.

Solution 2: Cherish it

Modulo the obstructions of *analytic* cocycles, get

$$\begin{array}{ccccc} \mathrm{H}^1(\Gamma, \mathcal{A}^{\times}/\mathbf{C}_p^{\times}) & \longrightarrow & \Lambda & & \\ \cap & & \cap & & \\ \mathrm{H}^1(\Gamma, \mathcal{M}^{\times}/\mathbf{C}_p^{\times}) & \longrightarrow & \mathrm{H}^2(\Gamma, \mathbf{C}_p^{\times}) & \longrightarrow & \frac{\mathbf{C}_p^{\times}}{\langle p\rangle} \times J_0(p)^2 \end{array}$$

Have

$$J_0(11) : y^2 + y = x^3 - x^2 - 10x - 20.$$

Obstruction of $\Theta_F^{\times}$ equals $(\alpha, P, 0)$ mod 11^{100}, where

$$\begin{cases} 121\alpha^2 + 206\alpha + 121 = 0, & \alpha \in \mathbf{Q}(\sqrt{-7}), \\ P = \left(\frac{-3-\sqrt{-7}}{2}, \frac{-3-\sqrt{-7}}{2}\right) & \in J_0(11). \end{cases}$$

The analytic argument of Zagier

Motivation

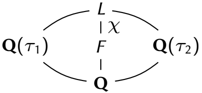
What about *traces* of CM singular moduli? Zagier (2002) makes modular generating series, and relates it to the theory of *Borcherds products*, which provide a morphism

$$\Psi_{\mathrm{Bor}} \;:\; M^{+,!}_{1/2}(\Gamma_0(4)) \;\longrightarrow\; \mathrm{H}^0(\mathrm{SL}_2(\mathbf{Z}), \mathcal{M}_\infty^\times).$$

Our belief in the existence of RM analogues of the *analytic* arguments in Gross–Zagier is strengthened by construction (Darmon-V. 2022) of a similar morphism

$$\Psi_{\mathrm{Rig}} \;:\; M^{+,!}_{1/2}(\Gamma_0(4p)) \;\longrightarrow\; \mathrm{H}^1(\Gamma, \mathcal{M}^\times).$$

Consider Hasse diagram:



Real analytic Hilbert Eisenstein series $E_s(z_1, z_2)$:

$$\sum_{[\mathfrak{a}]\in \mathrm{Cl}_F^+} \chi(\mathfrak{a})\,\mathrm{Nm}(\mathfrak{a})^{1+2s} \sum_{(m,n)\,\in\, \mathfrak{a}^2/\mathcal{O}_F^\times}{}' \frac{(mz_1+n)^{-1}(m'z_2+n')^{-1} y_1^s y_2^s}{|mz_1+n|^{2s}|m'z_2+n'|^{2s}}$$

Zagier (1983) then computes the Fourier expansion of

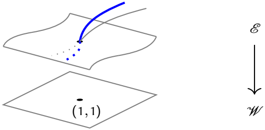
1. its diagonal restriction $E_s(z,z)$ (vanishes at $s=0$)
2. its analytic first order derivative with respect to s
3. its holomorphic projection, contained in $M_2(\mathrm{SL}_2(\mathbf{Z})) = \{0\}$.

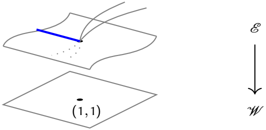
The first Fourier coefficient is of the form $\log \mathrm{Nm}(j(\tau_1) - j(\tau_2)) \;+\; \sum_q \mathrm{Int}_q \cdot \log(q)$.

The analytic argument (p-adic version)

Prove that $\Theta_p^\times[F, G]$ is algebraic, when $F = \langle 0, 1, 0\rangle$ is *degenerate*.

Relate $\log_p \Theta_p^\times[F, G] \longleftrightarrow$ Fourier coefficients of p-adic family through $E(\psi)$ over $\mathbf{Q}(\sqrt{D_G})$
$\longleftrightarrow$ p-adic deformations of Galois representation $\rho = 1 \oplus \psi$.





[Darmon-Pozzi-V. 2021] *Diagonal restrictions of p-adic Eisenstein families*
[Darmon-Pozzi-V. 2022] *On the RM values of the Dedekind–Rademacher cocycle*

Using an idea of Hecke–Klingen–Siegel, this proof also yields algorithm to compute p-adic Gross–Stark units, developed further by Håvard Damm-Johnsen (Thesis 2024).

Example

Let $K = \mathbf{Q}(\sqrt{136})$ with $\mathrm{Cl}^+(K) \simeq \mathbf{Z}/4\mathbf{Z}$, and $p = 19$ inert in K. For $F = \langle 0, 1, 0\rangle$ and G of discriminant 136, Håvard computes (in less than 3 seconds) that up to precision 19^{50} we have

$$\Theta_{19}^\times[F, G] \quad \text{satisfies} \quad 361x^4 + 508x^3 + 310x^2 + 508x + 361 = 0,$$

which generates the narrow Hilbert class field over K.

Future directions

What are some promising avenues?

1. **Computational:** Many challenges lie ahead!
   - Systematic computations in all cases (ramified, non-coprime, split, ...)

[Darmon–V. 2021] RM singular moduli for $SL_2(\mathbf{Z}[1/p])$, where p is inert.
[Darmon–V. 2022] Stark–Heegner points for $SL_2(\mathbf{Z}[1/p])$, where p is inert.
[Guitart–Masdeu–Xarles 2021] Quaternion orders over totally real fields.

   - Orthogonal group $O(3,1)$ by Darmon–Lipnowski–Gehrmann (forthcoming). Other orthogonal groups / general construction?
   - Factorisations of quaternionic / orthogonal invariants?

2. **Theoretical:** Two great mysteries remain, corresponding to
   - (Analytic) How far can analytic arguments with p-adic families be pushed, relying on deformations of Artin representations? Can one construct p-adic Borcherds lifts?

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[Darmon–V. 2022] *Real quadratic Borcherds products*

   - (Geometric) What is the geometric meaning of RM singular moduli?